An Introduction to Splines

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Contents

The following notes provides some basic definitions and properties of splines in view of interpolation and approximation purposes. They are partly inspired by the book of J.-J. Risler [Ris92]. Other popular references include Farin [Far02] and Gallier [Gal00].

1 Spline functions and curves

Originally, splines where used in the manufacturing industry to design smooth curves with some energy minimizing property. Those were thin wooden strips (called "splines") through points laid out on the floor. They were actually used for designing ships, automobiles or aircraft. Their use goes back as early as the 1600s.

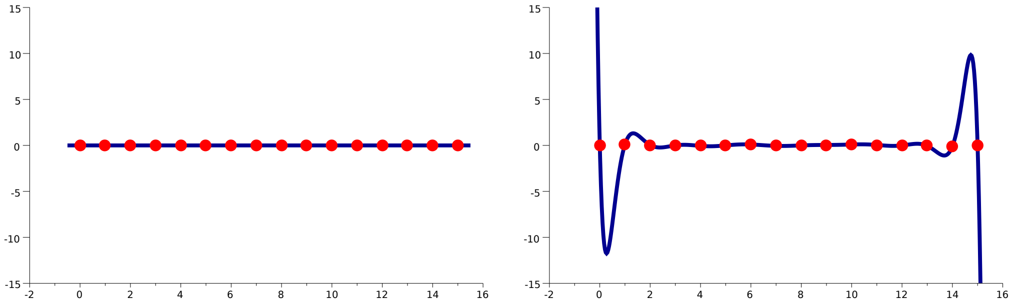

Figure 1: The Lagrange interpolating polynomial is highly sensible to the data point positions. Left, the Lagrange polynomial for horizontally aligned points. Right, the Lagrange polynomial for the slightly perturbed points.

Let $t_0 < t_1 < \ldots < t_n$ be a set of (real) parameters and let $y_0, y_1, \ldots, y_n$ be a set of (real) values. Suppose we want an explicit smooth function f **interpolating** the (t_i, y_i), i.e., such that $f(t_i) = y_i$ for $0 \le i \le n$.

The simplest is to consider a polynomial function. In particular, we can use the Lagrange interpolating polynomial (1795). It is defined as

$$\ell(t) = \sum_{i=0}^{n} y_i \left(\prod_{j=0, j\neq i}^{n} \frac{t - t_j}{t_i - t_j} \right)$$

One immediately checks that $\ell(t_i) = y_i$ for $0 \le i \le n$. This polynomial is the unique interpolating polynomial with degree at most n. Indeed, let f be an interpolating polynomial of degree at most n. Then $\ell - f$ has degree at most n and cancels at the $n+1$ parameters t_i. By d'Alembert fundamental Theorem of Algebra, $\ell = f$. The Lagrange interpolating polynomial is not always the prefered solution as some undesired oscillation effects, known as **Runge's phenomenon**, may appear. See Figure 1. One solution to cope with this phenomenon is to use higher degree polynomials with more control coefficients. However, manipulating high degree polynomials is computationally demanding and also subject to accuracy problems. Another prefered solution is to use a piecewise polynomial function of lower degree as described in the next section.

Exercise 1 *Suppose that instead of prescribing the values of a function f at $t_0, \ldots, t_n$, we impose the derivatives of f at t_0 up to order n. Can you find a degree n polynomial with the same derivatives? (Hint: you may use Taylor's theorem.)*

1.1 Cubic splines

Put $T = (t_0, t_1, \ldots, t_n)$ and let $\mathcal{S}_{3,T}$ be the set of C^2 piecewise polynomial functions over $[t_0, t_n]$ whose restriction to each interval $[t_i, t_{i+1})$ is a degree 3 polynomial[1]. This is a vector space (we can take linear combinations of such functions) with $\dim \mathcal{S}_{3,T} = n + 3$. To see this, note that each of the n polynomial pieces requires 4 coefficients and each of

[1]More generally, we denote by $\mathcal{S}_{k,T}$ the set of piecewise polynomials of degree k with continuity C^{k-1} at the parameters t_i.

the $n-1$ connections between them imposes 3 linear constraints, all independent. This leads to $4n-3(n-1)=n+3$ degrees of freedom.

For a function $f:\mathbb{R}\to\mathbb{R}$, let $f_+(t)=\max(f(t),0)$. A basis of $\mathcal{S}_{3,T}$ is given by the following set of $n+3$ functions:

$$S=\{\,(t-t_0)^k\,\}_{0\le k\le 3}\cup\{\,(t-t_i)_+^3\,\}_{1\le i\le n-1}$$

Indeed, those are easily seen to be independent functions in $\mathcal{S}_{3,T}$.

Lemma 1 (Interpolating property) *Given real numbers α,β and $y_0,y_1,\dots,y_n$ there exists a unique $f\in\mathcal{S}_{3,T}$ such that (1) : $f(t_i)=y_i$ for $0\le i\le n$ and (2) : $f'(t_0)=\alpha$ and $f'(t_n)=\beta$. In other words, f interpolates the y_i and its initial and final slopes are given by α and β respectively.*

Proof. Intuitively, conditions (1) and (2) define $n+3$ linear (affine) independent constraints on the coefficients of the polynomials. This is precisely the dimension of $\mathcal{S}_{3,T}$. $\square$

When $n=1$ and $T=(0,1)$, the set $\mathcal{S}_{3,T}$ is just the set of cubic polynomials over $[0,1]$. The interpolating property then states that there exists a unique cubic polynomials whose values and derivatives are fixed at 0 and 1. In this case $\dim\mathcal{S}_{3,T}=4$ and there is a convenient basis formed by the **Hermite** polynomials:

$$h_{00}(t):=(1+2t)(1-t)^2,\quad h_{01}(t):=t^2(3-2t),\quad h_{10}(t):=t(1-t)^2,\quad h_{11}(t):=t^2(t-1)$$

The unique spline $f\in\mathcal{S}_{3,T}$ defined by $f(0)=y_0, f(1)=y_1, f'(0)=\alpha$ and $f'(1)=\beta$ is given by

$$f=y_0h_{00}+y_1h_{01}+\alpha h_{10}+\beta h_{11}$$

This is an easy consequence of the following table 1. The functions in $\mathcal{S}_{3,T}$ are called

func F	$F(0)$	$F(1)$	$F'(0)$	$F'(1)$
h_{00}	1	0	0	0
h_{01}	0	1	0	0
h_{10}	0	0	1	0
h_{11}	0	0	0	1

Table 1: The values of the Hermite basis functions and their derivatives at the parameters $t=0$ and $t=1$.

splines in reference to the energy minimizing property of the original wood splines. Indeed, let α,β and $y_0,y_1,\dots,y_n$ be real numbers and let E be the set of C^2 functions ϕ over $[t_0,t_n]$ such that $\phi(t_i)=y_i$ for $0\le i\le n$ and $\phi'(t_0)=\alpha$ and $\phi'(t_n)=\beta$. We have

Lemma 2 (Energy minimizing property) *Among all $\phi \in E$ there exists a unique function f that minimizes $\int_{t_0}^{t_n} (\phi'')^2$. Moreover, $f \in \mathcal{S}_{3,T}$.*

Proof. From the interpolating property lemma there is a unique $f \in E \cap \mathcal{S}_{3,T}$. We first claim that for any $\phi \in E$ and any continuous piecewise linear function h with corners at the t_i, we have $\int_{t_0}^{t_n} (\phi - f)''h = 0$. Indeed, using integration by parts and cutting $[t_0, t_n]$ into the intervals $[t_i, t_{i+1}]$, we obtain:

$$\int_{t_0}^{t_n} (\phi - f)''h = [(\phi - f)'h]_{t_0}^{t_n} - \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} (\phi - f)'h'$$

From $\phi'(t_0) = f'(t_0) = \alpha$ and $\phi'(t_n) = f'(t_n) = \beta$ we conclude that $[(\phi - f)'h]_{t_0}^{t_n} = 0$. Because h' is a constant over each $[t_i, t_{i+1}]$, we also have $\int_{t_i}^{t_{i+1}} (\phi - f)'h' = [(\phi - f)h']_{t_i}^{t_{i+1}}$ which is also zero as $\phi(t_i) = f(t_i)$.

We can know write

$$\int_{t_0}^{t_n} (\phi'')^2 = \int_{t_0}^{t_n} ((\phi - f)'' + f'')^2 = \int_{t_0}^{t_n} ((\phi - f)'')^2 + \int_{t_0}^{t_n} (f'')^2 + 2\int_{t_0}^{t_n} (\phi - f)''f''$$

But the last integral cancels out by the above claim. It follows that $\int_{t_0}^{t_n} (\phi'')^2$ is minimal if and only if the continuous function $(\phi - f)''$ is zero everywhere. We infer that $\phi = f$ because these two functions have the same value and derivative at t_0. $\square$

1.2 B-splines

From the interpolation property we know that given a set of $n + 1$ values and given an initial and final slope, there exists a unique interpolating spline in $\mathcal{S}_{3,T}$. This spline is a linear combination of the S basis elements whose coefficients can be found by solving a set of linear equations. The exact formula for those coefficients is not simple, though. It would be nice to have another basis of $\mathcal{S}_{3,T}$ whose linear combinations are somehow related to the interpolation property. Ideally, if $\{b_i\}_{0 \le i \le n+2}$ is such a basis, we would hope that $f(t) = \sum_i y_i b_i(t)$ satisfies $f(t_i) = y_i$ (omitting the extremal slope conditions). This would imply $b_i(t_j) = \delta_{ij}$, where the usual Kronecker symbol δ_{ij} is zero whenever $i \neq j$ and $\delta_{ii} = 1$. A natural choice would be to take b_i with a bell shape and support in $[t_{i-1}, t_{i+1}]$. In particular, taking into account the C^2 continuity, we should have $b_i(t_i) = 1$ and $b_i'(t_i) = b_i(t_{i+1}) = b_i'(t_{i+1}) = b_i''(t_{i+1}) = 0$. However, there is no degree 3 polynomial satisfying those conditions on the interval $[t_i, t_{i+1}]$. Hence, our ideal and natural conditions can not be fulfilled. Defining a convenient spline basis appears to be a non trivial task.

A common basis, often used in CAD systems, is given by the so called **B-splines**. Those basis functions have nice geometric properties even though the coefficients of a spline in the B-spline basis are not *exactly* given by its values at the t_is. The B-splines are recursively defined as follows. We first introduce six extra parameters[2] t_{-3}, t_{-2}, t_{-1} and

[2]The parameters t_i are also called **knots**. In the general definition of B-splines, we only require $t_i \le t_{i+1}$, rather than $t_i < t_{i+1}$, for $-3 \le i < n+3$. Having a knot t_i repeated r times allows to decrease the degree of continuity at t_i to C^{k-r}. See [Ris92].

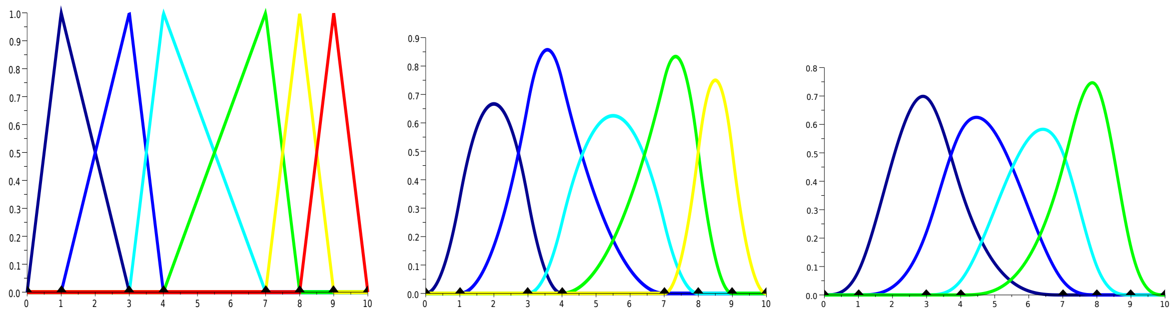

Figure 2: Right,the six degree 1 B-splines corresponding to the parameters (0,1,3,4,7,8,9,10). Middle, the five degree 2 B-splines corresponding to the same parameters. Right, the corresponding four degree 3 B-splines.

$t_{n+1}, t_{n+2}, t_{n+3}$ in addition to T satisfying $t_{-3} < t_{-2} < t_{-1} < t_0$ and $t_n < t_{n+1} < t_{n+2} < t_{n+3}$. Let $\omega_{i,k}$ be the unique affine function with $\omega_{i,k}(t_i) = 0$ and $\omega_{i,k}(t_{i+k}) = 1$. In other words[3],

$$\omega_{i,k}(t) = \frac{t - t_i}{t_{i+k} - t_i}$$

Let $B_{i,0}$ be the indicator function of the interval $[t_i, t_{i+1})$. Hence, $B_{i,0}(t) = 1$ over $[t_i, t_{i+1})$ and 0 elsewhere. For $k = 1, 2, 3$ and $-k \leq i \leq n-1$, we set

$$B_{i,k} = \omega_{i,k} B_{i,k-1} + (1 - \omega_{i+1,k}) B_{i+1,k-1} \tag{1}$$

By convention we also set $B_{i,k} = 0$ for $i < -k$ or $i > n-1$. A simple recursion on k shows that

Proposition 3 *$B_{i,k} \in \mathcal{S}_{k,T}$ is non-negative with support $[t_i, t_{i+k+1})$. Moreover the $B_{i,k}$, for $-k \leq i \leq n-1$, add up to one over $[t_0, t_n)$. In particular, the $B_{i,k}$ constitute a* ***partition of unity*** *over $[t_0, t_n)$.*

In most cases, the B-splines are continuous over $\mathbb{R}$ and we can extend the partition of unity over the closed interval $[t_0, t_n]$. Figure 2 shows the B-spline bases of degree 1,2 and 3 with respect to a same sequence of parameters. Figure 3 illustrates the fact that the sum of the B-splines of a fixed degree is one over a sub-interval. Beware that the definition of $B_{i,k}$ relies on the parameters $\{t_i\}_{-3 \leq i \leq n+3}$ even though we only consider its restriction to $[t_0, t_n)$. In order to prove that $\{B_{i,3}\}_{-3 \leq i \leq n-1}$ is indeed a basis of $\mathcal{S}_{3,T}$ we need the following

Lemma 4 *for $0 \leq k \leq 3$ and $t \in [t_0, t_n)$:*

$$(x - t)^k = \sum_{i=-k}^{n-1} c_{i,k}(t) B_{i,k}(x) \quad \textit{with} \quad c_{i,k}(t) = (t_{i+1} - t) \cdots (t_{i+k} - t)$$

[3] We set $\omega_{i,k} = 0$ whenever $t_i = t_{i+k}$.

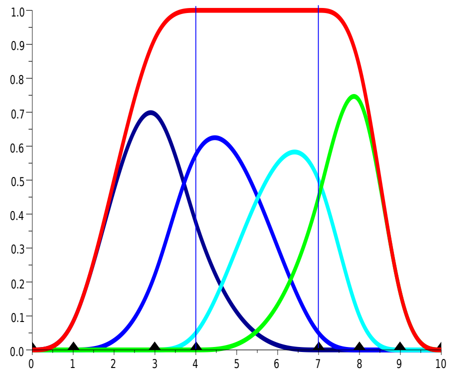

Figure 3: The same degree 3 B-splines as on Figure 2 together with their sum. Here, we have $t_{-3} = 0$, $t_{-2} = 1$,$t_{-1} = 3$, $t_0 = 4$, $t_1 = 7$, $t_2 = 8$, $t_3 = 9$, $t_4 = 10$. The sum of the B-spline functions is one over $[t_0, t_1]$.

Proof. This is true by definition for $k = 0$ (in this case $c_{i,k}$ is the unit constant function). Assume the lemma true up to degree $k-1 \geq 0$. Then, from the recursive definition of $B_{i,k}$ we have

$$\begin{aligned}\sum_{i=-k}^{n-1} c_{i,k}(t)B_{i,k}(x) &= \sum_{i=-k}^{n-1} c_{i,k}(t)(\omega_{i,k}(x)B_{i,k-1}(x) + (1-\omega_{i+1,k}(x))B_{i+1,k-1}(x)) \\ &= c_{-k,k}(t)\omega_{-k,k}(x)B_{-k,k-1}(x) + c_{n-1,k}(t)(1-\omega_{n,k}(x))B_{n,k-1}(x) + \\ &\quad \sum_{i=1-k}^{n-1}\big(c_{i,k}(t)\omega_{i,k}(x) + c_{i-1,k}(t)(1-\omega_{i,k}(x))\big)B_{i,k-1}(x) \qquad \text{(E)}\end{aligned}$$

In this last right member (E), the first two terms are null since from our convention: $B_{-k,k-1} = B_{n,k-1} = 0$. We also have

$$\begin{aligned}&c_{i,k}(t)\omega_{i,k}(x) + c_{i-1,k}(t)(1-\omega_{i,k}(x)) = \\ &\qquad c_{i,k-1}(t)(t_{i+k}-t)\omega_{i,k}(x) + c_{i,k-1}(t)(t_i - t)(1-\omega_{i,k}(x)) = \\ &\qquad c_{i,k-1}(t)\left((t_{i+k}-t)\frac{x-t_i}{t_{i+k}-t_i} + (t_i - t)\frac{t_{i+k}-x}{t_{i+k}-t_i}\right) = c_{i,k-1}(t)\cdot(x-t)\end{aligned}$$

It follows that the last sum in (E) reduces to $\sum_{i=1-k}^{n-1}(x-t)c_{i,k-1}B_{i,k-1}$ which is $(x-t)^k$ by our recursion assumption. □

Proposition 5 *$B = \{B_{i,3}\}_{-3\leq i\leq n-1}$ is a basis of $\mathcal{S}_{3,T}$.*

Proof. Since B has $n+3 = \dim \mathcal{S}_{3,T}$ elements it suffices to prove that B spans $\mathcal{S}_{3,T}$, or equivalently that every element in the basis S is a linear combination of elements in B. By the preceding lemma, applied with $k = 3$ and $t = t_0$, this is true for the S basis element $(x-t_0)^3$. Differentiating $(x-t)^3$ three times with respect to t and using the same formula in the lemma we deduce that $(x-t_0)^k$ is also spanned by B for $k = 0, 1, 2$. We

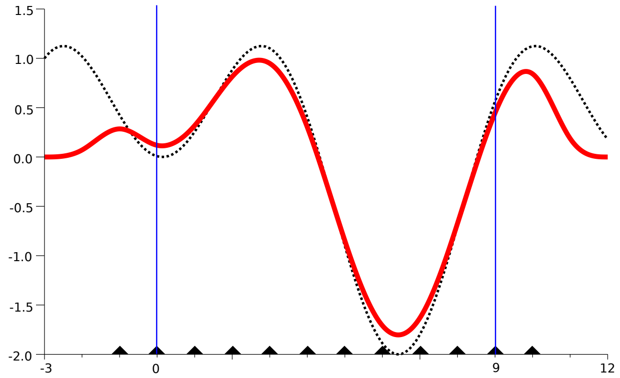

Figure 4: The graph of the function $f(t) = \sin(t/2) + \cos(t)$ (dotted curve) approximated by the (red) spline $\sum_{i=-3}^{8} f(t_i^*)B_{i,3}(t)$ with $t_i = i$, $-3 \le i \le 12$. Here, $t_i^* = i + 2$ for $-3 \le i \le 8$. The approximation is provably good over the interval $[t_0, t_n) = [0, 9]$.

finally claim that, for $1 \le j \le n-1$

$$(x - t_j)_+^3 = \sum_{i=j}^{n-1} c_{i,k}(t_j)B_{i,k}(x)$$

Indeed, for $x < t_j$ and $i \ge j$, Proposition 3 tells that $B_{i,k}(x) = 0$. It follows that both members are zero in the above equality. For $x \ge t_j$, this equality is just the same as in the lemma with $k = 3$ and $t = t_j$, noting that the missing terms in the sum are indeed null (either $c_{i,k}(t_j)$ or $B_{i,k}(x)$ is null). $\square$

Although the B-spline basis does not provide a simple formula for the coefficients of an interpolating spline, they have a nice behaviour with respect to approximation. Given *any* (not necessarily polynomial) C^2 function $f : [t_{-2}, t_{n+2}] \to \mathbb{R}$ we put

$$Sf(t) = \sum_{i=-3}^{n-1} f(t_i^*)B_{i,3}(t), \qquad \text{where } t_i^* = \frac{1}{3}(t_{i+1} + t_{i+2} + t_{i+3}) \tag{2}$$

Figure 4 shows an approximation of the function $f(t) = \sin(t/2) + \cos(t)$ over the interval $[0, 9]$ by the function Sf. If $h = \max_{0 \le i \le n-1}(t_{i+1} - t_i)$ is the maximal step size of T, and $\|f\|_I = \sup_{t \in I} |f(t)|$ is the C^0-norm of f over the interval I, we have

Theorem 6

$$\|f - Sf\|_{[t_0,t_n)} \le \frac{9}{2}h^2\|f''\|_{[t_{-2},t_{n+2})}$$

Proof. Let $t \in [t_i, t_{i+1})$ with $0 \le i \le n-1$. We want to prove that $|f(t) - Sf(t)| \le \frac{9}{2}h^2\|f''\|_I$, with $I = [t_{-2}, t_{n+2})$. Consider the tangent line φ of f at t. Its equation is given by

$$\varphi(x) = f(t) + f'(t)(x - t)$$

It is not hard to show that for the linear function φ we have $S\varphi = \varphi$ (see next exercise). In particular, $S\varphi(t) = \varphi(t) = f(t)$. We thus have

$$f(t) - Sf(t) = S\varphi(t) - Sf(t) = \sum_{j=i-k}^{i} (\varphi(t_j^*) - f(t_j^*))B_{j,3}(t)$$

The range of indices in the last summation is due to $B_{j,3}(t) = 0$ for $j > i$ or $j < i - k$, since we assumed $t \in [t_i, t_{i+1})$. By Taylor formula,

$$f(t_j^*) = f(t) + f'(t)(t_j^* - t) + \frac{f''(\xi)}{2!}(t_j^* - t)^2 = \varphi(t_j^*) + \frac{f''(\xi)}{2!}(t_j^* - t)^2 \text{ for some } t_j^* < \xi < t$$

Whence,

$$|\varphi(t_j^*) - f(t_j^*)| \leq |\frac{f''(\xi_j)}{2!}(t_j^* - t)^2| \leq \frac{\|f''\|_I}{2!}(t_j^* - t)^2$$

Note that for $-3 \leq j \leq n-1$, the inequalities $t_{-2} \leq t_{j+1} \leq t_j^* \leq t_{j+3} \leq t_{n+2}$ implies $\xi_j \in I$. Since $t_{j+1} \leq t_j^* \leq t_{j+3}$ and $t_i \leq t < t_{i+1}$, we have for $i - 3 \leq j \leq i$:

$$|t_j^* - t| \leq \max(|t_{i-3}^* - t_{i+1}|, |t_i^* - t_i|) \leq \max(|t_{i-2} - t_{i+1}|, |t_{i+3} - t_i|) \leq 3h.$$

We conclude that $|\varphi(t_j^*) - f(t_j^*)| \leq \frac{\|f''\|_I}{2!} 9h^2$, and finally

$$|f(t) - Sf(t)| \leq \sum_{j=1-k}^{i} (\frac{\|f''\|_I}{2!} 9h^2)B_{j,3}(t) \leq \frac{9}{2}\|f''\|_I h^2 \sum_{j=1-k}^{i} B_{j,3}(t) \leq \frac{9}{2}\|f''\|_I h^2.$$

where the last inequality results from $\sum_{j=1-k}^{i} B_{j,3}(t) \leq \sum_{j=-3}^{n-1} B_{j,3}(t) = 1$ $\square$

Exercise 2 *Show that the approximation formula (2) reproduces affine functions. In other words, show that $S(at + b) = at + b$.*

1.3 Spline curves

A **spline curve** $\gamma : [t_0, t_n) \to \mathbb{R}^2$ in the plane is a parametrized curve whose coordinates are spline functions. Writing $\gamma(t) = (x(t), y(t))$ we thus have $x(t) = \sum_{i=-3}^{n-1} x_i B_{i,3}(t)$ and $y(t) = \sum_{i=-3}^{n-1} y_i B_{i,3}(t)$. Putting $p_i = (x_i, y_i)$ we can write $\gamma(t) = \sum_{i=-3}^{n-1} p_i B_{i,3}(t)$. The polyline $p_{-3}, p_{-2}, \ldots p_{n-1}$ is called the **control polygon** of the spline curve γ with respect to the B-spline basis $B = \{B_{i,3}\}_{-3 \leq i \leq n-1}$. Note that the curve γ is defined over the interval $[t_0, t_n)$, although the definition of the B-spline basis uses the parameters $(t_{-3}, t_{-2}, \ldots, t_{n+3})$. Indeed, the combination of points p_i in the sum $\sum_{i=-3}^{n-1} p_i B_{i,3}(t)$ is only meaningful when $\sum_{i=-3}^{n-1} B_{i,3}(t) = 1$, which is insured over $[t_0, t_n)$ by the partition of unity property in Proposition 3.

Exercise 3 *Let $P = (p_{-3}, p_{-2}, \ldots p_{n-1})$ be the control polygon of a spline curve $f : [t_0, t_n) \to \mathbb{R}^2$, i.e., $f = \sum_{i=-3}^{n-1} p_i B_{i,3}(t)$. A convex combination of the control points in P is any affine combination $\sum_{i=-3}^{n-1} w_i p_i$ where $w_i \geq 0$ and $\sum_{i=1}^{n} w_i = 1$. The **convex hull** of P is the set of its convex combinations.*

- *Show that the curve traced by f for $t \in [t_0, t_n)$ is contained in the convex hull of P.*
- *Show more precisely that for any $t \in [t_i, t_{i+1})$, $0 \le i \le n-1$, the point $f(t)$ is in the convex hull of $\{p_{i-3}, p_{i-2}, p_{i-1}, p_i\}$.*

(Hint: you may use Proposition 3.)

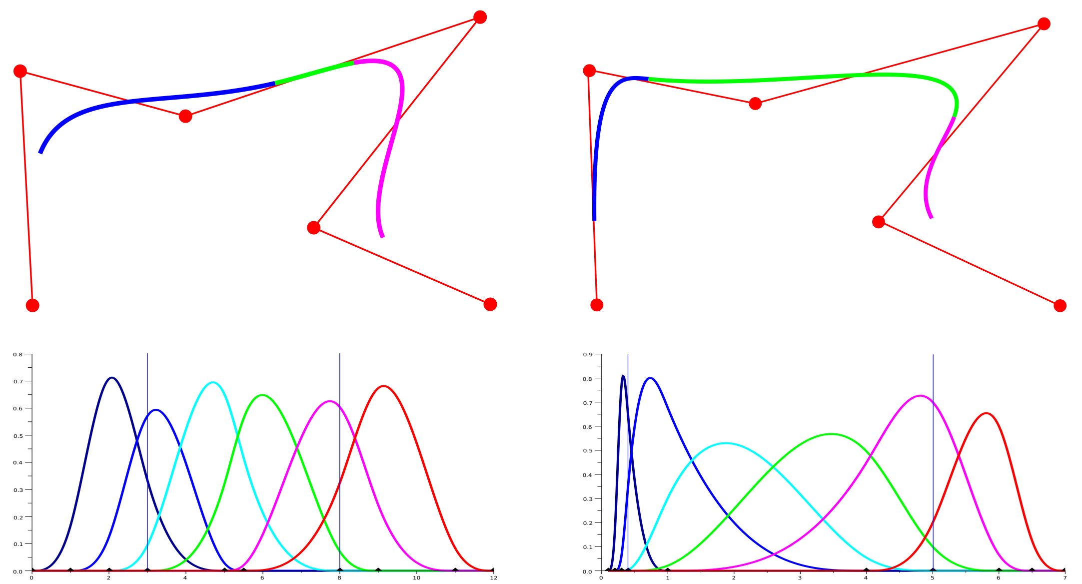

Figure 5: Example of degree 3 spline curves. The control polygon is shown in red. Each spline segment, i.e., the restrictions of the spline curve to the intervals $[t_i, t_{i+1})$, $0 \le i \le n-1$, is shown with a distinct color. Here $n = 3$. The figure emphasizes the influence of the choice of the parameters t_i. The sequence of parameters is $(0, 1, 2, 3, 5, 5.5, 8, 9, 11, 12)$ for the upper left figure and $(.1, .2, .3, .4, 1, 4, 5, 6, 6.5, 7)$ for the upper right figure. The lower figures show the corresponding B-spline bases (the B-spline colors are not related to the upper spline curve colors).

De Boor-Cox algorithm. There is a nice geometric construction that allows to draw a point of a spline from its control polygon using a ruler. The construction is due to De Boor-Cox for spline curves defined with B-splines and by De Casteljau when Bernstein polynomials are used in place of B-splines (see Section 1.4).

Proposition 7 *Let $t \in [t_j, t_{j+1})$ with $0 \le j \le n-1$ and let $\gamma = \sum_{i=-3}^{n-1} p_i B_{i,3}$ be a cubic spline curve. We can write*

$$\gamma(t) = \sum_{i=j-3}^{j} p_i^0 B_{i,3}(t) = \sum_{i=j-2}^{j} p_i^1 B_{i,2}(t) = \sum_{i=j-1}^{j} p_i^2 B_{i,1}(t) = p_j^3$$

where

$$p_i^0 = p_i \qquad \text{and} \qquad p_i^{r+1} = \omega_{i,3-r}(t) p_i^r + (1 - \omega_{i,3-r}(t)) p_{i-1}^r \tag{3}$$

and $\omega_{i,3-r}(t) = \frac{t - t_i}{t_{i+3-r} - t_i}$.

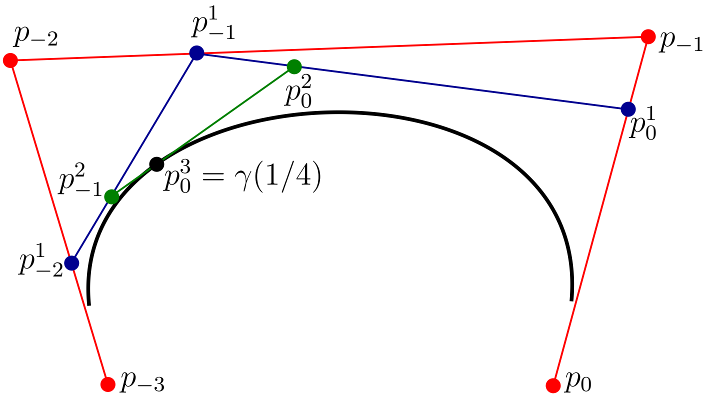

Figure 6: A degree 3 spline curve (thick black) defined by the parameters $(-0.3, -0.2, -0.1, 0, 1, 1.1, 1.2, 1.3)$ and the control polygon $(p_{-3}, p_{-2}, p_{-1}, p_0)$. In particular the curve is defined over the interval $[t_0, t_1) = [0, 1)$. The construction of $\gamma(1/4)$ is highlighted.

Note that p_i^{r+1} actually depends on the parameter t. We dropped the parameter to emphasize the geometric construction.

Proof. From the recursive formula (1) we can write for $k = 1, 2, 3$:

$$\begin{aligned}
\sum_{i=j-k}^{j} p_i^{3-k} B_{i,k} &= \sum_{i=j-k}^{j} p_i^{3-k}(\omega_{i,k} B_{i,k-1} + (1-\omega_{i+1,k})B_{i+1,k-1}) \\
&= \sum_{i=j-k}^{j} p_i^{3-k}\omega_{i,k} B_{i,k-1} + \sum_{i=j-k+1}^{j+1} p_{i-1}^{3-k}(1-\omega_{i,k})B_{i,k-1} \\
&= \sum_{i=j-k+1}^{j} (p_i^{3-k}\omega_{i,k} + p_{i-1}^{3-k}(1-\omega_{i,k}))B_{i,k-1} \\
&\quad + p_{j-k}^{3-k}\omega_{j-k,k} B_{j-k,k-1} + p_j^{3-k}(1-\omega_{j+1,k})B_{j+1,k-1}.
\end{aligned}$$

By Proposition 3, $B_{j-k,k-1}(t) = B_{j+1,k-1}(t) = 0$ for $t \in [t_j, t_{j+1})$. It follows that the last equality reduces to

$$\sum_{i=j-k}^{j} p_i^{3-k} B_{i,k} = \sum_{i=j-k+1}^{j} (p_i^{3-k}\omega_{i,k} + p_{i-1}^{3-k}(1-\omega_{i,k}))B_{i,k-1}.$$

Whence, putting $r = 3-k$, $p_i^{r+1} = p_i^r \omega_{i,3-r} + p_{i-1}^r(1-\omega_{i,3-r})$. □

Equation (3) expresses p_i^{r+1} as a barycenter of p_i^r and p_{i-1}^r with respective weights $\alpha = \omega_{i,3-r}(t)$ and $1-\alpha$. In particular, p_i^{r+1} belongs to the segment $p_i^r p_{i-1}^r$. We also note that the computation of $\gamma(t)$ is **local** in the sens that it only depends on the four points $p_{j-3}, p_{j-2}, p_{j-1}, p_j$. Figure 6 illustrates the geometric construction of a spline curve point.

1.4 Bernstein polynomials and Bézier curves

A particular case occurs for the B-spline basis of $\mathcal{S}_{3,T}$ when $T = (t_0, t_1)$, i.e. $n = 1$ and t_{-3}, t_{-2}, t_{-1} tend toward t_0 while $t_{n+1}, t_{n+2}, t_{n+3}$ tend toward t_1. If we further take $t_0 = 0$ and $t_1 = 1$, the recursion formula (1) becomes for $k = 1, 2, 3$ and $-k \le i \le 0$,

$$B_{i,k}(t) = tB_{i,k-1}(t) + (1-t)B_{i+1,k-1}(t)$$

This recursion easily solves to $B_{i,k}(t) = \binom{k}{i+k} t^{i+k}(1-t)^{-i}$. Putting $b_{j,k} = B_{j-k,k}$ for $0 \le j \le k$, we obtain

$$b_{j,k}(t) = \binom{k}{j} t^j (1-t)^{k-j} \tag{4}$$

In particular, we have from the binomial expansion theorem

$$1 = (t + (1-t))^k = \sum_{j=0}^{k} b_{j,k}(t) \tag{5}$$

The $b_{j,k}$ are called the **Bernstein polynomials**, named after the mathematician Sergei Natanovich Bernstein (1880–1968). Figure 7 shows the Bernstein bases of respective degree $k = 3$ and $k = 4$. A simple derivation of (4) shows that

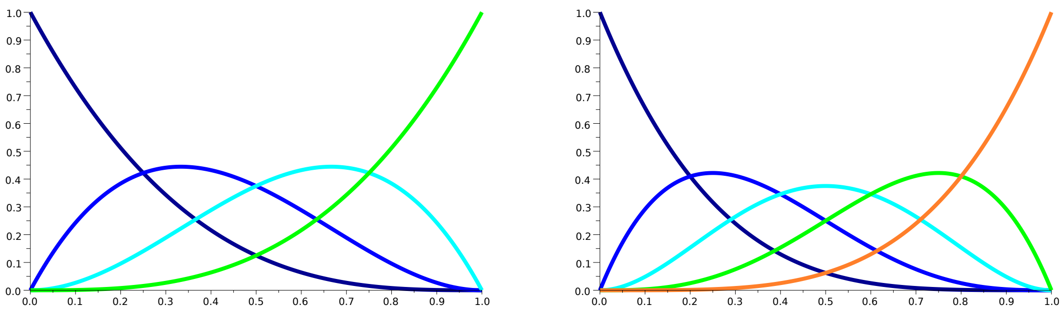

Figure 7: The Bernstein polynomials of degree 3 (left) and 4 (right).

$$b'_{j,k} = k(b_{j-1,k-1} - b_{j,k-1}) \tag{6}$$

It is intended that $b_{j,k} \equiv 0$ whenever $j < 0$ or $j > k$.

Exercise 4 *Prove equations* (4) *and* (6)*.*

Exercise 5 *Find the coefficients of the identity function in the Berstein Basis. In other words find c_0, c_1, c_2, c_3 such that $t = \sum_{j=0}^{3} c_j b_{j,3}(t)$. (Hint: you may use Exercise 2.)*

A **Bézier curve**

$$\gamma(t) = \sum_{j=0}^{3} p_j b_{j,3}(t)$$

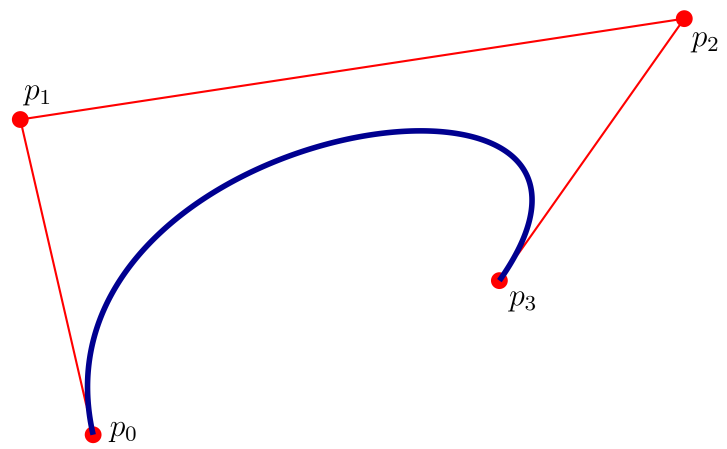

Figure 8: A degree 3 Bézier curve.

is a spline curve defined with Bernstein polynomials. Since $b_{j,k}(0) = \delta_{0,j}$ and $b_{j,k}(1) = \delta_{k,j}$, we have $\gamma(0) = p_0$ and $\gamma(1) = p_3$. Using the above derivation formula (6), we obtain

$$\gamma'(t) = 3\sum_{j=0}^{3} p_j(b_{j-1,2}(t) - b_{j,k2}(t)) = 3\sum_{j=0}^{2}(p_{j+1} - p_j)b_{j,2}$$

In particular, $\gamma'(t) = 3(p_1 - p_0)$ and $\gamma'(1) = 3(p_3 - p_2)$. It follows that the first and last segment of the control polygon (p_0, p_1, p_2, p_3) are tangent to γ at p_0 and p_3 respectively (See Figure 8). The De Boor-Cox algorithm simplifies for a Bézier curve and Equation 3 becomes (after shifting the indices by 3):

$$\begin{cases} p_i^0 & = & p_i, \quad 0 \le i \le 3 \\ p_i^{r+1} & = & tp_i^r + (1-t)p_{i-1}^r, \quad 0 \le r \le 2,\ 0 \le i \le 2 - r \end{cases}$$

This last algorithm is attributed to De Casteljau.

1.5 Knot insertion

Consider the set of splines $\mathcal{S}_{3,T}$, i.e., the set of C^2 functions whose restriction to each interval of the sequence of parameters $T = (t_0, t_1, \dots, t_n)$ is a cubic polynomial. Let $t_0 \le \tau \le t_n$ be an extra parameter and put $\hat{T} = T \cup \{\tau\}$. Then, every interval of $\hat{T}$ is contained in an interval of T. It follows that $\mathcal{S}_{3,T} \subset \mathcal{S}_{3,\hat{T}}$. We now consider extra parameters $t_{-3} < t_{-2} < t_{-1} < t_0$ and $t_n < t_{n+1} < t_{n+2} < t_{n+3}$ to define the B-spline bases $B = \{B_{i,3}\}_{-3 \le i \le n-1}$ and $\hat{B} = \{\hat{B}_{i,3}\}_{-3 \le i \le n}$ of $\mathcal{S}_{3,T}$ and $\mathcal{S}_{3,\hat{T}}$ respectively. Note that the number of B-splines in $\hat{B}$ is one more than in B. From what we just saw, every spline $\sum_i c_i B_{i,3} \in \mathcal{S}_{3,T}$ can be written

$$\sum_{i=-3}^{n-1} c_i B_{i,3} = \sum_{j=-3}^{n} \hat{c}_j \hat{B}_{j,3}$$

for some uniquely defined coefficients $\hat{c}_j$. Extending to the coordinates of a spline curve $\sum_i p_i B_{i,3}$, we have

$$\sum_{i=-3}^{n-1} p_i B_{i,3} = \sum_{i=-3}^{n} \hat{p}_i \hat{B}_{i,3}$$

for some uniquely defined control points $\hat{p}_i$. The same relation holds not only for degree 3 splines but for any degree k splines. The expression of $\hat{p}_i$ in terms of the control points p_i is recorded in the following

Proposition 8 (Boehm's algorithm) *Every spline curve* $\sum_{i=-k}^{n-1} p_i B_{i,k}$ *can be written* $\sum_{i=-k}^{n} \hat{p}_i \hat{B}_{i,k}$ *in the* $\hat{B}$ *basis of* $\mathcal{S}_{k,\hat{T}}$ *where*

$$\hat{p}_i = \begin{cases} p_i & \text{if } \; t_{i+k} \leq \tau \\ \omega_{i,k}(\tau) p_i + (1 - \omega_{i,k}(\tau)) p_{i-1} & \text{if } \; t_i < \tau < t_{i+k} \\ p_{i-1} & \text{if } \; t_i \geq \tau \end{cases} \tag{7}$$

If $t_j < \tau < t_{j+1}$ for some $j \in [0, n-1]$, the condition $t_{i+k} \leq \tau$ reduces to $i \leq j - k$, while $t_i < \tau < t_{i+k}$ translates to $j - k + 1 \leq i \leq j$ and $t_i \geq \tau$ translates to $i \geq j + 1$. However, the formulation in the proposition remains valid even when τ is equal to some t_j (See footnote 2). Figures 9 and 10 illustrate the insertion of new parameters in the parameter sequence of spline.

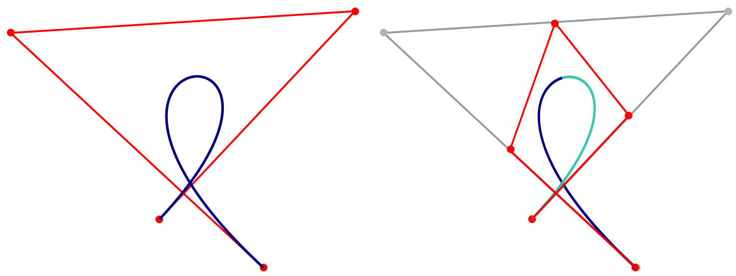

Figure 9: Left, a degree 3 spline defined by a control polygon with four vertices. Right, inserting a parameter adds a control point and splits the spline into two spline segments.

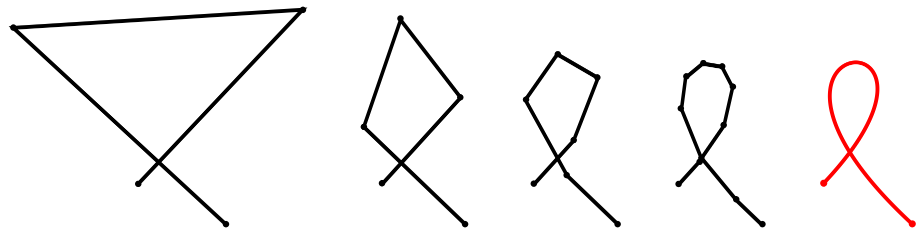

Figure 10: Successive insertions of parameters. The resulting sequence of control polygons converges toward the corresponding spline.

Proof. We prove the proposition by induction on the degree k of the splines. Let us put $\gamma(x) = \sum_{i=-k}^{n-1} p_i B_{i,k}(x)$ Thanks to the recursive definition (1) of B-splines we can write for $t_0 < x < t_n$.

$$\begin{aligned}
\gamma(x) &= \sum_{i=-k}^{n-1} p_i(\omega_{i,k}(x)B_{i,k-1}(x) + (1-\omega_{i+1,k}(x))B_{i+1,k-1}(x)) \\
&= \sum_{i=-k}^{n-1} p_i\omega_{i,k}(x)B_{i,k-1}(x) + \sum_{i=1-k}^{n} p_{i-1}(1-\omega_{i,k}(x))B_{i,k-1}(x) \\
&= \sum_{i=1-k}^{n-1} (\omega_{i,k}(x)p_i + (1-\omega_{i,k}(x))p_{i-1})B_{i,k-1}(x)
\end{aligned}$$

The indices in the last summation are justified by $B_{-k,k-1}(x) = B_{n,k-1}(x) = 0$. Putting $p_i^1 = \omega_{i,k}(x)p_i + (1-\omega_{i,k}(x))p_{i-1}$ we thus have $\gamma(x) = \sum_{i=1-k}^{n-1} p_i^1 B_{i,k-1}(x)$. Applying the induction hypothesis at order $k-1$, we then write

$$\gamma(x) = \sum_{i=1-k}^{n-1} \widehat{(p_i^1)}\hat{B}_{i,k-1}(x)$$

Where $\widehat{(p_i^1)}$ is given as a function of the p_i^1 according to (7). On the other hand, define

$$\hat{\gamma}(x) = \sum_{i=-k}^{n} \hat{p}_i\hat{B}_{i,k}$$

with $\hat{p}_i$ defined by (7). Thanks to the recursive equation (1) for order $k-1$, we obtain similarly as above

$$\hat{\gamma}(x) = \sum_{i=1-k}^{n} (\hat{p}_i)^1\hat{B}_{i,k-1}(x)$$

with $(\hat{p}_i)^1 = \hat{\omega}_{i,k}(x)\hat{p}_i + (1-\hat{\omega}_{i,k}(x))\hat{p}_{i-1}$. Hence, it suffices to show that $\widehat{(p_i^1)} = (\hat{p}_i)^1$ to conclude, as required for the induction, that $\gamma = \hat{\gamma}$. We first consider i such that $t_i < \tau < t_{i+k-1}$. On the one hand,

$$\begin{aligned}
\widehat{(p_i^1)} &= \omega_{i,k-1}(\tau)p_i^1 + (1-\omega_{i,k-1}(\tau))p_{i-1}^1 \\
&= \omega_{i,k-1}(\tau)(\omega_{i,k}(x)p_i + (1-\omega_{i,k}(x))p_{i-1}) \\
&\quad + (1-\omega_{i,k-1}(\tau))(\omega_{i-1,k}(x)p_{i-1} + (1-\omega_{i-1,k}(x))p_{i-2}) \\
&= \omega_{i,k-1}(\tau)\omega_{i,k}(x)p_i + \big(\omega_{i,k-1}(\tau)(1-\omega_{i,k}(x)) + (1-\omega_{i,k-1}(\tau))\omega_{i-1,k}(x)\big)p_{i-1} \\
&\quad + (1-\omega_{i,k-1}(\tau))(1-\omega_{i-1,k}(x))p_{i-2}
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
(\hat{p}_i)^1 &= \hat{\omega}_{i,k}(x)\hat{p}_i + (1-\hat{\omega}_{i,k}(x))\hat{p}_{i-1} \\
&= \hat{\omega}_{i,k}(x)(\omega_{i,k}(\tau)p_i + (1-\omega_{i,k}(\tau))p_{i-1}) \\
&\quad + (1-\hat{\omega}_{i,k}(x))(\omega_{i-1,k}(\tau)p_{i-1} + (1-\omega_{i-1,k}(\tau))p_{i-2}) \\
&= \hat{\omega}_{i,k}(x)\omega_{i,k}(\tau)p_i + \big(\hat{\omega}_{i,k}(x)(1-\omega_{i,k}(\tau)) + (1-\hat{\omega}_{i,k}(x))\omega_{i-1,k}(\tau)\big)p_{i-1} \\
&\quad + (1-\hat{\omega}_{i,k}(x))(1-\omega_{i-1,k}(\tau))p_{i-2}
\end{aligned}$$

Since $\hat{T}$ has one extra parameter τ with $t_i < \tau < t_{i+k-1}$, we have $\hat{t}_i = t_i$ and $\hat{t}_{i+k} = t_{i+k-1}$, whence $\hat{\omega}_{i,k} = \omega_{i,k-1}$. We then leave it to the reader to check that $f(\tau)g(x) = f(x)g(\tau)$ for $(f,g) = (\omega_{i,k-1}, \omega_{i,k})$ or $(\omega_{i,k-1}, 1-\omega_{i,k})$ or $(1-\omega_{i,k-1}, \omega_{i-1,k})$ or $(1-\omega_{i,k-1}, 1-\omega_{i-1,k})$. It directly follows that $\gamma = \hat{\gamma}$. The other cases when $t_i \geq \tau$ or $\tau \geq t_{i+k-1}$ can be treated in a similar way. $\square$

Exercise 6 *Deduce from the proposition how to express the B-splines $B_{j,k}$ in the $\hat{B}$ basis. (Hint: you can write $B_{j,k} = \sum_{i=-k}^{n-1} c_i B_{i,k}$ with $c_i = \delta_{i,j}$)*

1.6 Bézier curve subdivision

In Section 1.4, we introduced Bézier curves as special cases of splines. We can thus insert new parameters as for any spline following the previous section. However, after introducing one parameter the new spline is not expressed anymore as a Bézier curve. The left Figures 9 actually represents a Bézier curve whose spline basis are Bernstein polynomials. After inserting a parameter, the right Figures 9 represent the same spline in a spline basis that is no more composed of Bernstein polynomials. If we introduce 3 identical parameters (respectively k for a degree k Bézier curve) at the same place then it can be shown that the control polygon has 7 (respectively $2k+1$) control points whose median point lies on the spline curve. To see this, observe that the knot insertion procedure described in Proposition 8 is exactly the same as the first step of the De Boor-Cox construction described in Proposition 7. Indeed this first step can be written $\gamma(t) = \sum_{i=-3}^{n-1} p_i B_{i,3}(t) = \sum_{i=-2}^{n} p_i^1 B_{i,2}(t)$ Inserting the parameter τ three times thus amounts to compute $\gamma(\tau)$ as a new control point. In fact, inserting τ one more time will double this control point (to see this, you can just apply (7) as in Proposition 8). Starting with a Bézier curve $\gamma(t) = \sum_{i=0}^{3} p_i b_{i,3}(t) = \sum_{i=-3}^{0} p_{i+3} B_{i,3}(t)$ corresponding to the knot sequence $T = (0,0,0,0,1,1,1,1)$, we can thus write after inserting τ four times

$$\gamma(t) = \sum_{i=-3}^{4} q_i \tilde{B}_{i,3}(t) = \sum_{i=-3}^{0} q_i \tilde{B}_{i,3}(t) + \sum_{i=1}^{4} q_i \tilde{B}_{i,3}(t)$$

where $\{\tilde{B}_{i,3}\}_{-3\leq i\leq 4}$ is the B-spline basis for the knot sequence $\tilde{T} = (0,0,0,0,\tau,\tau,\tau,\tau,1,1,1,1)$ and $q_3 = q_4$. For $t \in [0,\tau)$, the partial sum $\sum_{i=-3}^{0} q_i \tilde{B}_{i,3}(t)$ only depends on the knot sequence $(0,0,0,0,\tau,\tau,\tau,\tau)$ and we can write

$$\gamma(t) = \sum_{i=-3}^{0} q_i B_{i,3}(\tau t)$$

where the $B_{i,3}$ corresponds to T, i.e. are Bernstein polynomials. Likewise, for $t \in [\tau, 1)$,we can write the partial sum $\sum_{i=1}^{4} q_i \tilde{B}_{i,3}(t)$ as

$$\gamma(t) = \sum_{i=1}^{4} q_i B_{i-4,3}(\frac{t-\tau}{1-\tau})$$

It follows that the Bézier curve γ has been split, after re-parametrization, into two Bézier curves with control polygons $(q_{-3}, q_{-2}, q_{-1}, q_0)$ and (q_1, q_2, q_3, q_4). See Figures 11 and 12 for an illustration.

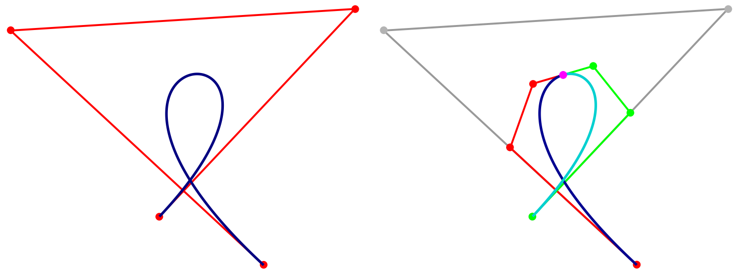

Figure 11: Left, the same spline has on left Figure 9. It is actually defined as degree 3 Bézier curve. Right, inserting a parameter 3 times at the same position adds 3 control points and the spline can be split into 2 Bézier curves by doubling the fourth control point.

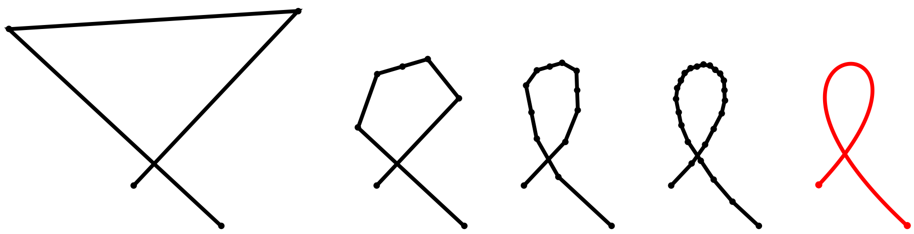

Figure 12: Successive splits. The resulting sequence of control polygons converges toward the corresponding spline.

2 Spline surfaces

2.1 Tensor product surfaces

Once we have defined one dimensional spline functions, a natural generalisation to two dimension is to consider products of such functions. Hence, if f and g are one dimensional functions defined over the respective intervals I_f and I_g, we naturally have a function $f \otimes g : I_f \times I_g \to \mathbb{R}$ defined by $f \otimes g(u, v) = f(u)g(v)$. Given two spaces of splines $\mathcal{S}_{3,T}$ and $\mathcal{S}_{3,T'}$ the set of functions we obtain by taking linear combinations of products $f \otimes g$, $f \in \mathcal{S}_{3,T}$ and $g \in \mathcal{S}_{3,T'}$, can be identified with the **tensor product** of spaces $\mathcal{S}_{3,T} \otimes \mathcal{S}_{3,T'}$. If B is a basis of $\mathcal{S}_{3,T}$ and B' is a basis of $\mathcal{S}_{3,T'}$, the set of functions $\{b \otimes b'\}_{b \in \mathcal{S}_{3,T}, b' \in \mathcal{S}_{3,T'}}$ is a basis of $\mathcal{S}_{3,T} \otimes \mathcal{S}_{3,T'}$. Considering the Bernstein basis polynomials (see Section 1.4), we obtain the **tensor product Bézier splines** :

$$f(u, v) = \sum_{i=0}^{3} \sum_{j=0}^{3} c_{ij} b_{i,3}(u) b_{j,3}(v)$$

where $c_{ij} \in \mathbb{R}$ are the **control coefficients**. Figure 13 shows the graph of some of the tensor product basis splines $b_{i,j}^{3,3}(u, v) = b_{i,3}(u) b_{j,3}(v)$. A simple summation, using Equation (5) shows that

$$\sum_{0 \le i,j \le 3} b_{i,j}^{3,3} = \sum_{i=0}^{3} \sum_{j=0}^{3} b_{i,3} b_{j,3} = 1$$

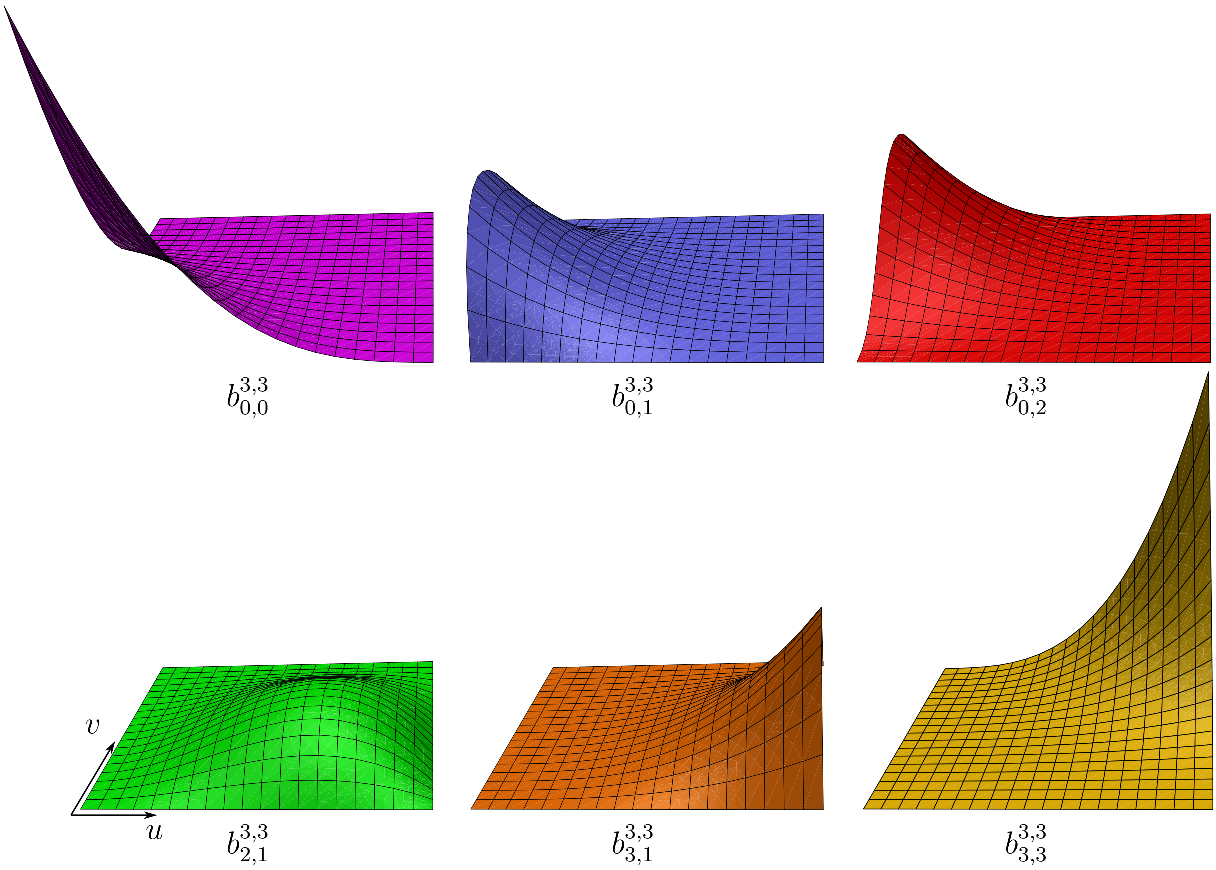

Figure 13: Not all of the 16 basis splines are represented. Note the simple symmetry between $b^{3,3}_{0,0}$ and $b^{3,3}_{3,3}$ or between $b^{3,3}_{0,2}$ and $b^{3,3}_{3,1}$. In fact any of the basis splines is symmetric to one of $b^{3,3}_{0,0}$, $b^{3,3}_{0,1}$ or $b^{3,3}_{2,1}$.

over the domain $[0,1] \times [0,1]$. We can thus take convex combinations of **control points** $p_{i,j} \in \mathbb{R}^3$ with coefficients $b^{3,3}_{i,j}$ to define a **tensor product Bézier patch**

$$P(u,v) = \sum_{i=0}^{3} \sum_{j=0}^{3} p_{ij} b_{i,3}(u) b_{j,3}(v)$$

The 16 control points of a Bézier patch of degree $(3,3)$ form a **control net** obtained by connecting the control points into a quadrangular grid as on Figure 14. Writing $p_j(u) = \sum_{i=0}^{3} p_{ij} b_{i,3}(u)$ we see that

$$P(u,v) = \sum_{j=0}^{3} p_j(u) b_{j,3}(v)$$

In other words, for a fixed u, the curve $v \mapsto P(u,v)$ is a Bézier curve with control points $p_j(u)$. This simple observation allows to compute $P(u,v)$ by applying the De Casteljau algorithm five times: four times to compute $p_0(u), p_1(u), p_2(u), p_3(u)$ and one more time to compute $P(u,v)$.

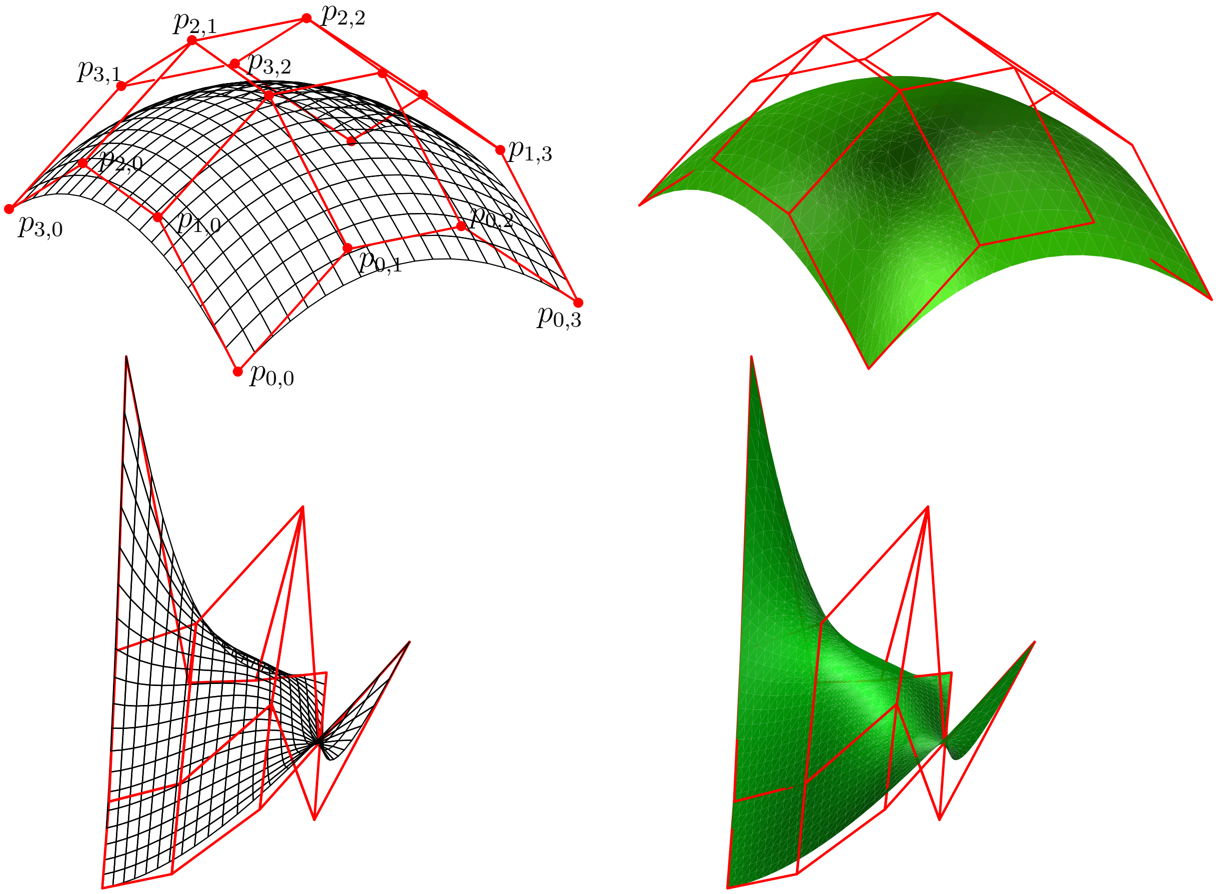

Figure 14: Two control nets with their rendered Bézier patch.

2.2 Hermite Patches

Using the Hermite basis functions (see Section 1.1) in place of the Bernstein polynomials we obtain a **Hermite patch**. Putting $H_{2i+j} = h_{ij}$, i.e., $H_0 = h_{00}$, $H_1 = h_{01}$, $H_2 = h_{10}$ and $H_3 = h_{11}$ this leads to the form:

$$P(u, v) = \sum_{i=0}^{3} \sum_{j=0}^{3} q_{ij} H_i(u) H_j(v)$$

for $q_{ij} \in \mathbb{R}^3$. With the help of Table 1 we easily compute:

$$q_{ij} = \begin{cases} P(i, j), & 0 \leq i, j \leq 1 \\ \dfrac{\partial P}{\partial u}(i-2, j), & 2 \leq i \leq 3, 0 \leq j \leq 1 \\ \dfrac{\partial P}{\partial v}(i, j-2), & 0 \leq i \leq 1, 2 \leq j \leq 3 \\ \dfrac{\partial^2 P}{\partial u\, \partial v}(i-2, j-2), & 2 \leq i, j \leq 3 \end{cases}$$

This simple relationship between the control points and the surface patch allows to easily approximate a given parametrized surface $M : [a, b] \times [c, d] \to \mathbb{R}^3$ by a set of Hermite patches. To do so, subdivide the parameter domain $[a, b] \times [c, d]$ into smaller rectangles $[a_i, b_i] \times [c_j, d_j]$ and estimate each surface piece $M([a_i, b_i] \times [c_j, d_j])$ by a Hermite patch defined by the piece corners $M(a_i, c_j)$, $M(b_i, c_j)$, $M(b_i, d_j)$, $M(a_i, d_j)$ and by the (estimated) derivatives $\frac{\partial M}{\partial u}$, $\frac{\partial M}{\partial v}$, $\frac{\partial^2 M}{\partial u \partial v}$ at those corners. The resulting union of Hermite patches will have continuous first order derivatives and continuous crossed derivative. It is a C^1 surface, though not C^2 in general. Figure 15 illustrates this approximation for a torus of revolution. Assuming that the torus is parametrized over $[0, 1] \times [0, 1]$, the parameter domain is divided into 16 pieces corresponding to the subdomains $[i/4, (i + 1)/4] \times [j/4, (j + 1)/4]$, $0 \leq i, j \leq 3$.

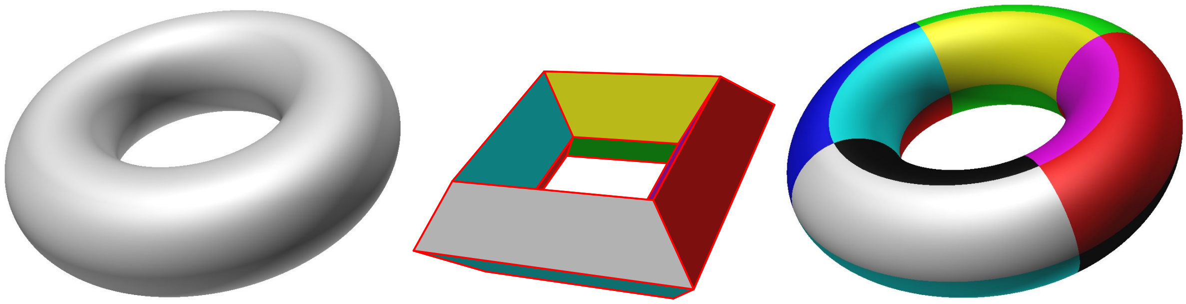

Figure 15: Left, a standard torus of revolution. Middle, a polyhedral approximation whose vertices are sampled on the standard torus. Right, each face is replaced by a Hermite patch with the same vertices using exact derivatives as computed on the standard torus.

2.3 Triangular Bézier patches

Tensor product surfaces have a quadrangular domain. If one wants to cover a surface with tensor product surfaces, the surface must be quadrangulated first as illustrated on Figure 16. It is sometimes more convenient to work with triangular patches as on

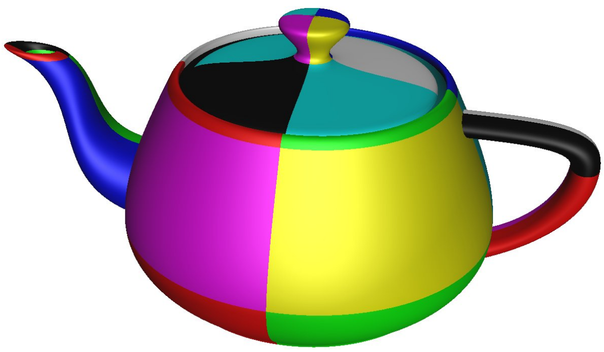

Figure 16: A quadrangulated teapot.

Figure 17. Those are parametrized surfaces defined over the standard triangle $\Delta = \{(r, s, t) \mid r + s + t = 1 \text{ and } r, s, t \geq 0\}$ with polynomial coordinates. Given a net of control points $\{P_{i,j,k}\}_{i+j+k=3}$, the corresponding triangular patch of degree 3 is given by

$$S(r, s, t) = \sum_{i+j+k=3} P_{i,j,k} b_{i,j,k}(r, s, t)$$

where b_{i+j+k} is the Bernstein polynomial (with 3 variables)

$$b_{i,j,k}(r,s,t) = \frac{3!}{i!j!k!} r^i s^j t^k$$

By the multinomial theorem, we have

$$1 = (r+s+t)^3 = \sum_{i+j+k=3} b_{i,j,k}(r,s,t)$$

As for spline curves and tensor product surfaces, it follows that a triangular patch is included in the convex hull of its control points.

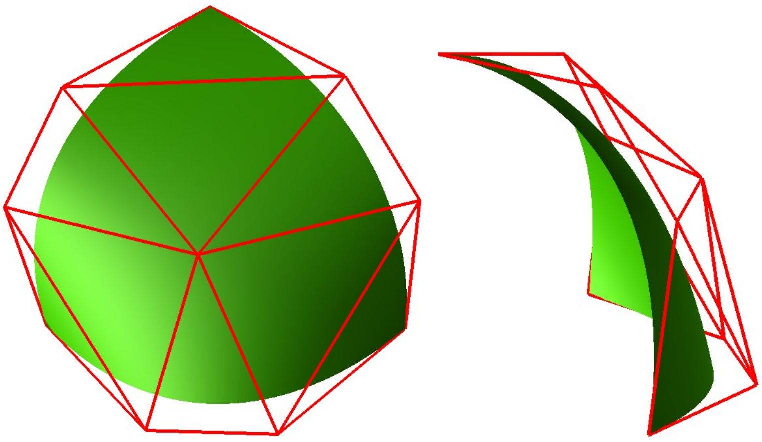

Figure 17: Two views of a triangular Bézier patch of degree 3 with its control net.

3 Splines and Polar Forms

In Section 1.2, B-splines were presented as a basis for $\mathcal{S}_{3,T}$ and defined by the recursive formula(1). All subsequent properties of B-splines where deduced from this formula. There is a more algebraically inclined way of introducing B-splines. The De Boor evaluation algorithm or the insertion algorithm are almost straightforward in this approach. On the other hand the presentation is a little more abstract. It relies on a classical relation between homogeneous polynomials and symmetric multilinear forms. An excellent introduction is provided by Seidel [Sei93].

3.1 Polar Forms

A map $f : \mathbb{R}^n \to \mathbb{R}$ is **multilinear** if it is linear in each argument, holding the other ones fixed, i.e. if $f(x_1, \ldots, \sum_i \lambda_i y_i, \ldots, x_n) = \sum_i \lambda_i f(x_1, \ldots, y_i, \ldots, x_n)$. Likewise, a map is **multiaffine** if it is affine in each argument, holding the other ones fixed. Recall that a map is **affine** if it preserves affine combinations[4]. Hence, $g : \mathbb{R}^n \to \mathbb{R}$ is multiaffine if

$$g(x_1, \dots, \sum_i \alpha_i y_i, \dots, x_n) = \sum_i \alpha_i g(x_1, \dots, y_i, \dots, x_n) \tag{8}$$

whenever $\sum_i \alpha_i = 1$. A map $h : \mathbb{R}^n \to \mathbb{R}$ is **symmetric** if its value does not change when permuting its arguments, i.e., for any permutation $\sigma \in \mathcal{S}_n$

$$h(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)})) = h(x_1, x_2, \dots, x_n)$$

To every monomial x^n we can associate the n-linear symmetric map $p_n : \mathbb{R}^n \to \mathbb{R}$, $(x_1, \dots, x_n) \mapsto \prod_{i=1}^n x_i$. We observe that $x^n = p_n(x, \dots, x)$. If $k \leq n$, we can also associate to x^k the n-affine symmetric map

$$e_k(x_1, \dots, x_n) = \frac{1}{\binom{n}{k}} \sum_{\{i_1, \dots, i_k\} \subset [n]} p_k(x_{i_1}, \dots, x_{i_k})$$

where $[n]$ is the set $\{1, \dots, n\}$ of integers. The e_k are usually called the **elementary symmetric polynomials**. We also observe that $x^k = e_k(x, x, \dots, x)$. Hence, every degree n polynomial $p(x) = \sum_{i=0}^n a_i x^i$ can be associated the n-affine symmetric map

$$\varphi = \sum_{i=0}^{n} a_n e_k \tag{9}$$

so that $p(x) = \varphi(x, \dots, x)$. We call φ the n-**affine polar form** associated to p.

Theorem 9 *Every polynomial p of degree at most n has a unique n-affine polar form, i.e., a unique n-affine symmetric map φ such that*

$$p = \varphi \circ \mathrm{diag}$$

where $\mathrm{diag} : \mathbb{R} \to \mathbb{R}^n$ *is the map* $\mathrm{diag}(x) = (\underbrace{x, \dots, x}_{n})$.

Proof. Equation (9) shows the existence of a polar map. The difficult part is to show uniqueness. We provide two proofs. The first proof gives an intrinsic and explicit formula for the polar form in terms of p. The second proof shows that the polar form establishes an isomorphism between polynomials of degree at most n and n-affine maps. The proof are somehow technical and are essentially here for the interested reader.

First proof. It suffices to show that any n-affine symmetric map $f : \mathbb{R}^n \to \mathbb{R}$ is entirely determined by its restriction to the diagonal $p_f = f \circ \mathrm{diag} : \mathbb{R} \to \mathbb{R}$. For this, we show that

$$n! f(x_1, \dots, x_n) = \sum_{I \subset [n]} (-1)^{n-|I|} |I|^n p_f\left(\frac{\sum_{i \in I} x_i}{|I|}\right) \tag{10}$$

[4]An affine combination of $y_1, \dots, y_k$ is any combination $\sum_i \alpha_i y_i$ with $\sum_i \alpha_i = 1$. Thus, a convex combination is an affine combination with non-negative coefficients.

We denote the right hand side as $F(x_1,\dots,x_n)$. Note that F only depends on p_f. We view $\frac{\sum_{i\in I} x_i}{|I|}$ as an affine combination of the x_i with weights $\frac{1}{|I|}$. By (8), we have

$$\begin{aligned} p_f\Big(\frac{\sum_{i\in I} x_i}{|I|}\Big) &= f\Big(\frac{\sum_{i\in I} x_i}{|I|},\dots,\frac{\sum_{i\in I} x_i}{|I|}\Big) \\ &= \frac{1}{|I|^n}\sum_{(i_1,\dots,i_n)\in I^n} f(x_{i_1},\dots,x_{i_n}) \end{aligned}$$

Reporting in (10), we obtain

$$F(x_1,\dots,x_n) = \sum_{I\subset[n]}(-1)^{n-|I|}\sum_{(i_1,\dots,i_n)\in I^n} f(x_{i_1},\dots,x_{i_n}) \tag{11}$$

Let $\alpha=(\alpha_1,\dots,\alpha_n)\in[n]^n$ denote a multi-index and let I_α be the set of distinct indices in α. Hence, for $n=5$ and $\alpha=(2,1,2,4,4)$, we have $I_\alpha=\{1,2,4\}$. We also define $S_\alpha=\{I\subset[n]\mid I_\alpha\subset I\}$. In our example, $S_\alpha=\big\{\{1,2,4\},\{1,2,3,4\},\{1,2,4,5\},\{1,2,3,4,5\}\big\}$. We finally write x_α for $(x_{\alpha_1},\dots,x_{\alpha_n})$. Equation (11) can now be written

$$F(x_1,\dots,x_n)=\sum_{\alpha\in[n]^n}\Big(\sum_{I\in S_\alpha}(-1)^{n-|I|}\Big)f(x_\alpha)$$

We now observe that the map $I\mapsto[n]\setminus I$ is a bijection between S_α and the power set $\mathcal{P}([n]\setminus I_\alpha)$, the set of all subsets of the complement of I_α in $[n]$. Unless a set is empty, its power set contains as many subsets of even and odd cardinal (this is an exercise). It follows that, unless α is a permutation of $[n]$,

$$\sum_{I\in S_\alpha}(-1)^{n-|I|}=0$$

Hence, using the fact that the set $\mathcal{S}_n$ of permutations of $[n]$ contains $n!$ permutations and using that f is symmetric, the equation for F reduces to

$$F(x_1,\dots,x_n)=\sum_{\alpha\in\mathcal{S}_n} f(x_\alpha)=n!f(x_1,\dots,x_n)$$

which was to be proven.

Second proof. By (8) and writing $x_i=(1-x_i).0+x_i.1$ we have for any n-affine map f:

$$\begin{aligned} f(x_1,\dots,x_i,\dots,x_n) &= (1-x_i)f(x_1,\dots,x_{i-1},0,x_{i+1},\dots,x_n) \\ &\quad + x_i f(x_1,\dots,x_{i-1},1,x_{i+1},\dots,x_n) \end{aligned}$$

Assuming that f is also symmetric, we can reorder the n parameters as we wish. With this mind, we can expand f as

$$\begin{aligned} f(x_1,\dots,x_n) &= \sum_{I\subset[n]}\Big(\prod_{i\in I}(1-x_i)\prod_{j\subset[n]\setminus I}x_j\Big)f(\underbrace{0,\dots,0}_{|I|},\underbrace{1,\dots,1}_{n-|I|}) \\ &= \sum_{k=0}^{n}\sum_{\substack{I\subset[n]\\|I|=k}}\Big(\prod_{i\in I}(1-x_i)\prod_{j\subset[n]\setminus I}x_j\Big)f(\underbrace{0,\dots,0}_{k},\underbrace{1,\dots,1}_{n-k}) \end{aligned}$$

Putting $r_k(x_1,\ldots,x_n) = \sum_{\substack{I\subset[n]\\|I|=k}} \left(\prod_{i\in I}(1-x_i)\prod_{j\subset[n]\setminus I} x_j\right)$ we obtain f as a linear combination of the n-affine symmetric maps r_k:

$$f = \sum_{k=0}^{n} f(\underbrace{0,\ldots,0}_{k},\underbrace{1,\ldots,1}_{n-k})r_k \tag{12}$$

It follows that the set of n-affine symmetric maps, viewed as a subspace of $\mathbb{R}[x_1,\ldots,x_n]$, has dimension at most $n+1$. It has in fact exactly this dimension since it contains the elementary symmetric polynomials that are clearly independent (you can use successive partial derivations to see this). But the set of univariate polynomials of degree at most n has dimension $n+1$ too. The linear map $\Phi : p \mapsto \varphi$ defined by (9) and the linear map $\Psi : f \mapsto f\circ \mathrm{diag}$ are thus inverse isomorphisms since we obviously have $\Psi\circ\Phi = Id$. □

3.2 Derivatives and Polar Forms

We will see that the derivatives of a polynomial can be computed from its polar form. We first need the following lemma whose easy proof is left to the reader. It essentially tells that for any affine map f, the map $f - f(0)$ is linear.

Lemma 10 *Let $f : \mathbb{R}\to\mathbb{R}$ be an affine map. Then,*

- *for any $a,b\in\mathbb{R}$, the value $f(b)-f(a)$ only depends on $b-a$, i.e., $f(b+h)-f(a+h) = f(b)-f(a)$ for all h.*
- *for any $\lambda, a\in\mathbb{R}$, we have $f(\lambda a) - f(0) = \lambda(f(a)-f(0))$.*

As a corollary, if $\varphi : \mathbb{R}^n\to\mathbb{R}$ is n-affine then

$$\varphi(x_1,\ldots,x_{n-1},x_n+h) - \varphi(x_1,\ldots,x_{n-1},x_n) = h\big(\varphi(x_1,\ldots,x_{n-1},1) - \varphi(x_1,\ldots,x_{n-1},0)\big)$$

We now state the relationship between the derivatives of a polynomial p of degree at most n and its n-affine polar form φ.

Proposition 11 *For any $0\le k\le n$ and any $x\in\mathbb{R}$:*

$$p^{(k)}(x) = \frac{n!}{(n-k)!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\varphi(\underbrace{x,\ldots,x}_{n-k},\underbrace{0,\ldots,0}_{k-j},\underbrace{1,\ldots,1}_{j}) \tag{13}$$

Proof. The proof is by recursion on k. We first prove the formula for $k=1$. We have

$$\begin{aligned}
\frac{p(x+h)-p(x)}{h} &= \frac{1}{h}(\varphi((x+h)^{(n)}) - \varphi(x^{(n)})) = \frac{1}{h}(\varphi(x+h,\ldots,x+h) - \varphi(x,\ldots,x))\\
&= \frac{1}{h}\sum_{j=1}^{n}\big(\varphi(\underbrace{x+h,\ldots,x+h}_{j},\underbrace{x,\ldots,x}_{n-j}) - \varphi(\underbrace{x+h,\ldots,x+h}_{j-1},\underbrace{x,\ldots,x}_{n-j+1})\big)\\
&= \sum_{j=1}^{n}\big(\varphi(\underbrace{x+h,\ldots,x+h}_{j-1},\underbrace{x,\ldots,x}_{n-j},1) - \varphi(\underbrace{x+h,\ldots,x+h}_{j-1},\underbrace{x,\ldots,x}_{n-j},0)\big)
\end{aligned}$$

Where the last equality is a consequence of the above corollary and the fact that φ is symmetric. Taking the limit on both sides, we obtain

$$p'(x) = n(\varphi(\underbrace{x, \dots, x}_{n-1}, 1) - \varphi(\underbrace{x, \dots, x}_{n-1}, 0))$$

We now assume that (13) is true up to order k. We then compute

$$p^{(k+1)}(x) = (p^{(k)})'(x) = \frac{n!}{(n-k)!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\frac{\mathrm{d}}{\mathrm{d}x}\varphi(\underbrace{x, \dots, x}_{n-k}, \underbrace{0, \dots, 0}_{k-j}, \underbrace{1, \dots, 1}_{j})$$

Viewing $\varphi(\underbrace{x, \dots, x}_{n-k}, \underbrace{0, \dots, 0}_{k-j}, \underbrace{1, \dots, 1}_{j})$ as a symmetric $(n-k)$-affine map and applying the case $k = 1$ we have, using the notation $x^{(k)}$ for a parameter x repeated k times:

$$\frac{\mathrm{d}}{\mathrm{d}x}\varphi(x^{(n-k)}, 0^{(k-j)}, 1^{(j)}) = (n-k)\big(\varphi(x^{(n-k-1)}, 0^{(k-j)}, 1^{(j+1)}) - \varphi(x^{(n-k-1)}, 0^{(k-j+1)}, 1^{(j)})\big)$$

Reporting in the above expression for $p^{(k+1)}(x)$, we get

$$\begin{aligned}
p^{(k+1)}(x) &= \frac{n!(n-k)}{(n-k)!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\big(\varphi(x^{(n-k-1)}, 0^{(k-j)}, 1^{(j+1)}) - \varphi(x^{(n-k-1)}, 0^{(k-j+1)}, 1^{(j)})\big) \\
&= \frac{n!}{(n-k-1)!}\Big(\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\varphi(x^{(n-k-1)}, 0^{(k-j)}, 1^{(j+1)}) \\
&\quad + \sum_{j=0}^{k}(-1)^{k-j+1}\binom{k}{j}\varphi(x^{(n-k-1)}, 0^{(k-j+1)}, 1^{(j)})\Big) \\
&= \frac{n!}{(n-k-1)!}\Big(\sum_{j=1}^{k+1}(-1)^{k-j+1}\binom{k}{j-1}\varphi(x^{(n-k-1)}, 0^{(k-j+1)}, 1^{(j)}) \\
&\quad + \sum_{j=0}^{k}(-1)^{k-j+1}\binom{k}{j}\varphi(x^{(n-k-1)}, 0^{(k-j+1)}, 1^{(j)})\Big) \\
&= \frac{n!}{(n-k-1)!}\Big(\sum_{j=1}^{k}(-1)^{k-j+1}\Big(\binom{k}{j} + \binom{k}{j-1}\Big)\varphi(x^{(n-k-1)}, 0^{(k-j+1)}, 1^{(j)}) \\
&\quad + \varphi(x^{(n-k-1)}, 1^{(k+1)}) + (-1)^{k+1}\varphi(x^{(n-k-1)}, 0^{(k+1)})\Big) \\
&= \frac{n!}{(n-k-1)!}\Big(\sum_{j=0}^{k+1}(-1)^{k+1-j}\binom{k+1}{j}\varphi(x^{(n-k-1)}, 0^{(k+1-j)}, 1^{(j)})\Big)
\end{aligned}$$

In the last equality we used the recursive formula for binomial coefficients $\binom{k}{j} + \binom{k}{j-1} = \binom{k+1}{j}$. $\square$

Corollary 12 *Let p and q be two polynomials of degree at most n and let φ and ψ their respective n-polar form. Given $x \in \mathbb{R}$ and $k \leq n$, p and q have the same derivatives up to order k at x if and only if*

$$\varphi(x^{(n-k)}, 0^{(k-j)}, 1^{(j)}) = \psi(x^{(n-k)}, 0^{(k-j)}, 1^{(j)}), \quad \text{for } 0 \leq j \leq k \tag{14}$$

PROOF. The proof is by recursion on k. The base case $k = 0$ is a tautology. Suppose that the corollary is true up to order k. If p and q have equal derivatives up to order $k+1$ at x then they obviously have equal derivatives up to order k and, by our recursion assumption, they satisfy the relations (14) in the corollary. We consider the values $u_{k+1,j} = \varphi(x^{(n-k-1)}, 0^{(k+1-j)}, 1^{(j)}) - \psi(x^{(n-k-1)}, 0^{(k+1-j)}, 1^{(j)})$ with $0 \leq j \leq k+1$ as $k+2$ unknowns. Each equality in (14) can be written as a linear equation in these unknowns. Indeed, thinking of the last parameter x as $(1-x).0 + x.1$, we derive

$$u_{k,j} = (1-x)u_{k+1,j} + xu_{k+1,j+1}, \qquad 0 \leq j \leq k$$

This leads to $k+1$ equations that are easily seen to be linearly independent. The condition $p^{(k+1)}(x) = q^{(k+1)}(x)$ translates by (13) to another independent linear equation. We thus have $k+2$ independent equations for $k+2$ unknowns and conclude that all the unknowns must be null, i.e. that (14) is true at order $k+1$. $\square$

Corollary 13 *We use the same notations as in the previous corollary. We also consider $2k$ real numbers $y_1 \leq y_2 \leq \cdots \leq y_k < y_{k+1} \leq y_{k+2} \cdots \leq y_{2k}$. Then, p and q have the same derivatives up to order k at x, if and only if*

$$\varphi(x^{(n-k)}, y_{j+1}, y_{j+2}, \ldots, y_{j+k}) = \psi(x^{(n-k)}, y_{j+1}, y_{j+2}, \ldots, y_{j+k}), \quad \text{for } 0 \leq j \leq k \tag{15}$$

PROOF. We just need to show that the conditions (14) and (15) are equivalent. Assuming (15), we have, after re-ordering the parameters

$$(\varphi - \psi)(x^{(n-k)}, y_{j+2}, \ldots, y_{j+k}, y_{j+1}) = 0 \text{ and } (\varphi - \psi)(x^{(n-k)}, y_{j+2}, \ldots, y_{j+k}, y_{j+k+1}) = 0$$

It follows that the affine map $u_1 \mapsto (\varphi - \psi)(x^{(n-k)}, y_{j+2}, \ldots, y_{j+k}, u_1)$ is null. By a simple recursion we conclude that for all $u_1, \ldots, u_k$, $(\varphi - \psi)(x^{(n-k)}, u_1 \ldots, u_k) = 0$. In particular, choosing $u_i \in \{0, 1\}$ implies (14). The same argument holds starting with the sequence $(\underbrace{0, \ldots, 0}_{k}, \underbrace{1, \ldots, 1}_{k})$ in place of $(y_1, y_2, \ldots, y_{2k})$. $\square$

3.3 From Splines to Polar Forms

We turn back to the set of splines $\mathcal{S}_{k,T}$ where $T = (t_0, t_1, \ldots, t_n)$. To keep the presentation simple, we assume that T is a strictly increasing sequence. We recall that $\mathcal{S}_{k,T}$ is the set of C^{k-1} piecewise polynomial functions over $[t_0, t_n]$ whose restriction to each interval $[t_i, t_{i+1})$

is a degree k polynomial. We also consider $2k-2$ extra parameters[5]

$$t_{1-k} \leq t_{1-k} \leq \cdots \leq t_0 < t_n \leq t_{n+1} \leq \cdots t_{n+k-1}$$

Let $\pi \in \mathcal{S}_{k,T}$ be a spline and denote by $p_i : \mathbb{R} \to \mathbb{R}$ the polynomial with the same restriction as π over $[t_i, t_{i+1})$, for $0 \leq i \leq n-1$. Let φ_i be the k-affine polar form of p_i.

Lemma 14 *p_i is entirely determined by $\varphi_i(t_{j+1}, \ldots, t_{j+k})$ for $i-k \leq j \leq i$.*

PROOF. We already know from (12) that φ_i, hence p_i, is determined by the values $\varphi_i(\underbrace{0,\ldots,0}_{\ell},\underbrace{1,\ldots,1}_{k-\ell})$ for $0 \leq \ell \leq k$. Those values are themselves determined by the $\varphi_i(t_{j+1}, \ldots, t_{j+k})$. The proof of this last claim is analogous to the proof of Corollary 15. □

Lemma 15 *For $1 \leq i \leq n-1$, the C^{k-1} continuity of π at t_i is equivalent to*

$$\varphi_i(t_{j+1}, \ldots, t_{j+k}) = \varphi_{i+1}(t_{j+1}, \ldots, t_{j+k}), \quad i-k \leq j \leq i-1$$

PROOF. This is precisely condition (15) at order $k-1$ applied to $x = t_i$ and $(y_1, y_2, \ldots, y_{2k-2}) = (t_{i-k+1}, t_{i-k+2}, \ldots, t_{i-1}, t_{i+1}, \ldots, t_{i+k-1}, t_{i+k})$. □

From the two previous lemmas, we see that the spline $\pi \in \mathcal{S}_{k,T}$ is determined by $n+k$ coefficients $c(t_{j+1}, \ldots, t_{j+k})$ for $-k \leq j \leq n-1$ such that the polar form φ_i of the restriction p_i of π to $[t_i, t_{i+1})$ satisfies

$$\varphi_i(t_{j+1}, \ldots, t_{j+k}) = c(t_{j+1}, \ldots, t_{j+k}), \qquad i-k \leq j \leq i$$

The $c(t_{j+1}, \ldots, t_{j+k})$ are called the **control coefficients** of π. We simply denote them by $c_j = c(t_{j+1}, \ldots, t_{j+k})$. It is not difficult to see, starting for instance from (12), that π depends linearly on its control coefficients c_j. We thus have a surjective linear map $\mathcal{S}_{k,T} \to \mathbb{R}^{n+k}$ that sends any spline π to its the vector of control coefficients $\{c_j\}$. Since $\dim \mathcal{S}_{k,T} = n+k$, this map is actually a bijection. Let $\{\beta_{i,k}\}_{-k \leq i \leq n-1}$ be the reciprocal image of the canonical basis of $\mathbb{R}^{n+k}$. It know turns out that

Proposition 16 *$\beta_{i,k}$ is the restriction to $[t_0, t_n)$ of the B-spline $B_{i,k}$ defined in Section 1.2. In other words, the spline whose control coefficients are $c_j = \delta_{i,j}$ is the jth B-spline.*

As a consequence, we can write

$$\pi = \sum_{i=-k}^{n-1} c_j B_{i,k}$$

Here we find the same expression for a spline curve as the one given in Section 1.3. The control coefficients are thus the control points in the one dimensional case.

[5]In Section 1.2, we introduced $2k$ extra parameters. The two more parameters t_{-k}, t_{n+k} are only needed to define the first and last B-splines in the corresponding basis. But the restrictions of those two B-splines to the interval $[t_0, t_n)$ is actually independent of t_{-k}, t_{n+k} . In the more local point of view of polar forms, those two parameters are not needed.

Proof of the proposition. In the next section, it is shown that the evaluation procedure for polar forms applied to the spline π with control coefficients $\{c_j\}$ and the evaluation procedure of Proposition 7 applied to $\sum_{i=-k}^{n-1} c_j B_{i,k}$ are the same. It follows that $\pi(t) = \sum_{i=-k}^{n-1} c_j B_{i,k}$, i.e. that $\sum_{i=-k}^{n-1} c_j \beta_{i,k} = \sum_{i=-k}^{n-1} c_j B_{i,k}$. Since this is true for any vector of coefficients it must be that $\{B_{i,k}\}$ and $\{\beta_{i,k}\}$ are the same bases. $\square$

3.4 Evaluation and Insertion Revisited

Evaluation. As above, we consider a spline π of degree k whose restriction p_i to $[t_i, t_{i+1})$ has polar form φ_i. Let c_j be the control coefficients of π and suppose that for some $t \in [t_i, t_{i+1})$ we want to compute $\pi(t)$ in terms of the c_j. To this end, we set for $0 \le r \le k$:

$$c_j^r = \varphi_i(\underbrace{t, \ldots, t}_{r}, t_{j+1}, \ldots, t_{j+k-r}) \quad \text{with } i-k+r \le j \le i$$

In particular, $c_j^0 = \varphi_i(t_{j+1}, \ldots, t_{j+k}) = c_j$ and $c_i^k = \varphi_i(\underbrace{t, \ldots, t}_{k}) = p_i(t) = \pi(t)$. We can compute c_i^k recursively from the c_j^0 by expressing t as an affine combination of t_{j+k-r} and t_j. Indeed, we can write

$$t = \omega_{j,k-r}(t) t_{j+k-r} + (1 - \omega_{j,k-r}(t)) t_j$$

where we recall that $\omega_{j,\ell}(t) = \frac{t-t_j}{t_{j+\ell}-t_j}$. It follows that

$$\begin{aligned}
c_j^{r+1} &= \varphi_i(\underbrace{t, \ldots, t}_{r+1}, t_{j+1}, \ldots, t_{j+k-r-1}) \\
&= \varphi_i(\underbrace{t, \ldots, t}_{r}, \omega_{j,k-r} t_{j+k-r} + (1-\omega_{j,k-r}) t_j, t_{j+1}, \ldots, t_{j+k-r-1}) \\
&= \omega_{j,k-r} \varphi_i(\underbrace{t, \ldots, t}_{r}, t_{j+k-r}, t_{j+1}, \ldots, t_{j+k-r-1}) \\
&\quad + (1 - \omega_{j,k-r}(t)) \varphi_i(\underbrace{t, \ldots, t}_{r}, t_j, t_{j+1}, \ldots, t_{j+k-r-1}) \\
&= \omega_{j,k-r} c_j^r + (1 - \omega_{j,k-r}(t)) c_{j-1}^r
\end{aligned}$$

This is exactly the De Boor-Cox recursion of Proposition 7! Figure 18 illustrates the evaluation algorithm in terms of polar form.

Knot insertion. Suppose that we want to introduce a new knot $\tau \in [t_i, t_{i+1})$. π is now cut into $n+1$ polynomial pieces $\{q_j\}_{0 \le j \le n}$ and the knot sequence becomes $(t'_{1-k}, \ldots, t'_{n+k}) = (t_{1-k}, \ldots, t_0, \ldots, t_i, \tau, t_{i+1}, \ldots, t_{n+k})$. In other words

$$t'_j = \begin{cases} t_j & \text{if } 1-k \le j \le i \\ \tau & \text{if } j = i+1 \\ t_{j-1} & \text{if } i+2 \le j \le n-k \end{cases}$$

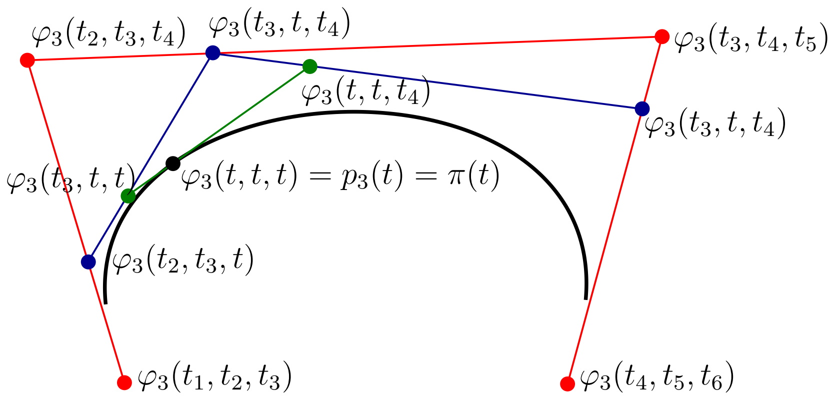

Figure 18: The evaluation algorithm for a degree 3 spline at $t\in[t_3,t_4)$.

We obviously have

$$q_j=\begin{cases} p_j & \text{if } 0\le j\le i-1\\ \text{the restriction of } p_i \text{ to } [t_i,\tau) & \text{if } j=i\\ \text{the restriction of } p_i \text{ to } [\tau,t_{i+1}) & \text{if } j=i+1\\ p_{j-1} & \text{if } i+2\le j\le n\end{cases}$$

The question is to compute the control coefficients d_j of the *same* spline π with respect to the new knot sequence. If $j+k<i$ then the sequences $(t'_{j+1},\dots,t'_{j+k})$ and $(t_{j+1},\dots,t_{j+k})$ are identical and because we are evaluating the same polar forms, we have

$$d_j=d_j(t'_{j+1},\dots,t'_{j+k})=c_j(t_{j+1},\dots,t_{j+k})=c_j$$

Likewise, if $j+1>i$ the sequences $(t'_{j+1},\dots,t'_{j+k})$ and $(t_j,\dots,t_{j+k-1})$ are identical and we have $d_j=c_{j-1}$. Otherwise the sequence $(t'_{j+1},\dots,t'_{j+k})$ contains τ. Writing $\tau=\omega'_{j,k+1}(\tau)t'_{j+k+1}+(1-\omega'_{j,k+1}(\tau))t'_j$ similarly as above and noting that $\omega'_{j,k+1}=\omega_{j,k}$ (because $t'_{j+k+1}=t_{j+k}$), we obtain

$$\begin{aligned} d_j(t'_{j+1},\dots,t'_{j+k}) &= \omega_{j,k}(\tau)d_j(t'_{j+1},\dots,\hat{\tau},\dots,t'_{j+k+1})\\ &\quad+(1-\omega_{j,k}(\tau))d_j(t'_j,\dots,\hat{\tau},\dots,t'_{j+k})\end{aligned}$$

where $\dots,\hat{\tau},\dots$ stands for the removal of τ. In other words $d_j=\omega_{j,k}(\tau)c_j+(1-\omega_{j,k}(\tau))c_{j-1}$. We find again the Boehm's algorithm as in Proposition 8!

Some other course materials can be found on the web:

- http://ibiblio.org/e-notes/Splines/Intro.htm
- http://www.cs.mtu.edu/~shene/COURSES/cs3621/NOTES/
- http://www-ljk.imag.fr/membres/Nicolas.Szafran/ENSEIGNEMENT/MASTER2/

References

[Far02] Gerald Farin. *Curves and surfaces for CAGD. A practical guide.* Morgan Kaufmann, fifth edition edition, 2002.

[Gal00] Jean Gallier. *Curves and surfaces in geometric modeling. Theory and Algorithms.* Morgan Kaufmann, 2000.

[Ris92] J.-J. Risler. *Mathematical Methods for CAD.* Cambridge University Press, 1992.

[Sei93] Hans-Peter Seidel. An introduction to polar forms. *IEEE Computer Graphics and Applications*, 13(1):38–46, 1993.