CR13: Computational Topology
Exercises #4
Due October 19th

1. Let G be a connected graph that is not a tree. Recall that the orientable (resp. non-orientable) genus $g(G)$ (resp. $\tilde{g}(G)$) of a graph is the smallest genus of an orientable (resp. non-orientable) surface on which it embeds. Show that they satisfy:

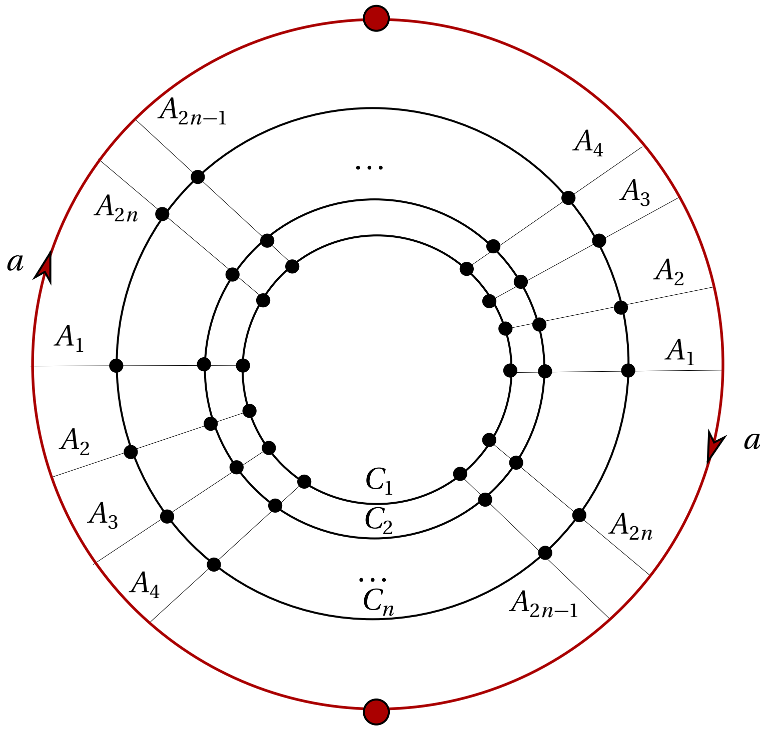
$$\tilde{g}(G) \leq 2g(G) + 1.$$

Solution: Let S be a surface of minimal orientable genus on which G can be embedded. Then this embedding is cellular, for otherwise one could embed G on a surface of smaller genus by replacing the faces that are not disks with disks. Now, pick one of the disks of this cellular embedding, and replace it with a Möbius band. The resulting surface $\tilde{S}$ is non-orientable (because it contains a Möbius band), and the embedding of G on $\tilde{S}$ is non-cellular, but it can be made cellular by adding one edge, cutting the Möbius band into a single disk. Then, comparing the Euler characteristics of S and $\tilde{S}$, we obtain that $g(\tilde{S}) \leq 2g(G) + 1$, and thus $\tilde{g}(G) \leq 2g(G) + 1$.

Let G_n be the family of graphs in Figure 1. The A_i are edges that wrap around and identify opposite points.

2. Show that for every n, G_n embeds in the projective plane.

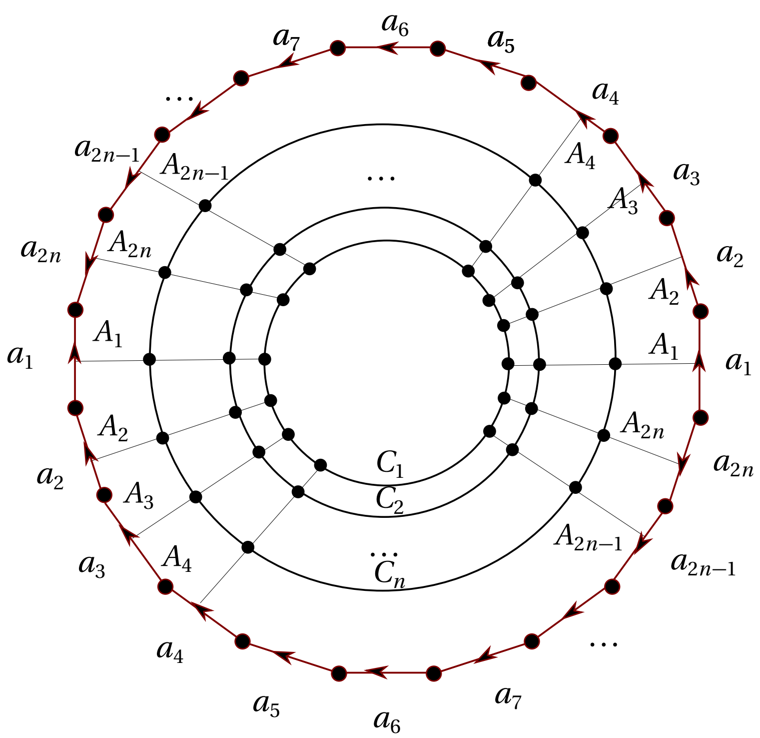
Solution: The following figure shows an embedding of G_n on the projective plane.



3. Show that for every n, G_n embeds on the orientable surface of genus n.

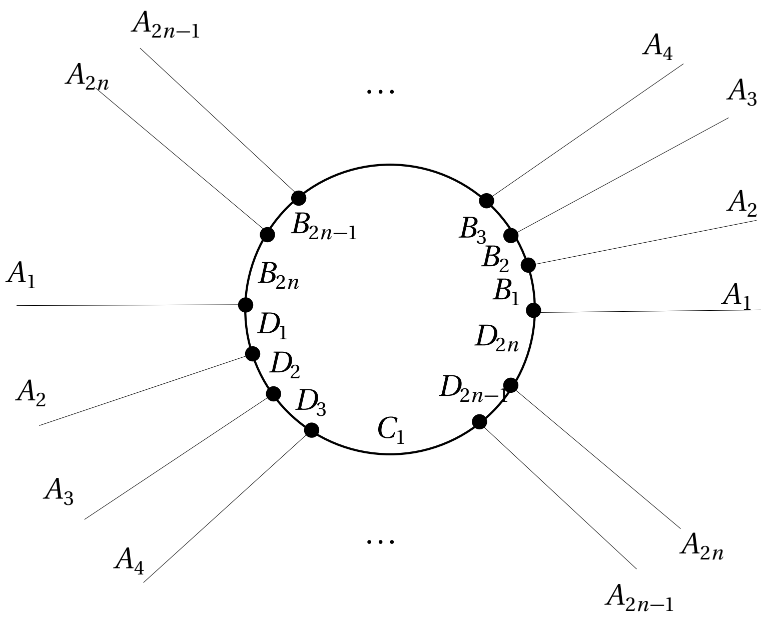
   *Solution:*The following figure shows an embedding of G_n on a surface with polygonal scheme $a_1a_2a_3\dots a_{2n-1}a_{2n}\bar{a}_1\bar{a}_2\bar{a}_3\dots \bar{a_{2n-1}}\bar{a_{2n}}$. This surface is orientable (no edges of the polygonal scheme are identified with the same orientation), and has 1 vertex, 1 face and $2n$ edges, thus its Euler characteristing is $2-2n$ and its genus is n.

   Henceforth, we assume that G_n is embedded on an orientable surface S_n of genus g. The subgraph K_n is defined in Figure 2, and inherits an embedding on S_n from the embedding of G_n.



4. Show that if C_1 bounds a face that is a disk, then S has genus at least n. *Hint: Compute the faces of* K_n.

   Solution: The graph K_n, being a minor of G_n, inherits naturally an embedding on S_n from the embedding of G_n, in which C_1 also bounds a face that is a disk. Since all the vertices of K_n are on C_1 and have degree 3, the fact that C_1 is a face forces the cyclic orderings of the edges $A_1, \dots A_{2n}$ around the vertices to be as in Figure 2, since they can not enter that disk. This embedding has $6n$ edges and $4n$ vertices. For the faces, there is one inside the disk, of boundary $B_1 \dots B_{2n} D_1 \dots D_{2n}$ (with the notations of the picture below), and following a boundary, we see that there is a single other one, of boundary $A_1 B_1 A_2 D_2 A_3 B_3 A_4 \dots A_{2n-1} B_{2n-1} A_{2n} D_{2n} A_1 D_2 A_2 \dots A_{2n} B_{2n}$ (without taking care of the edge orientations). If the embedding is cellular, the Euler characteristic gives $g = n$. Otherwise, adding edges to make it cellular only increases the genus, and thus $g \geq n$.



5. Show that if C_1 bounds a disk D (but not necessarily a face of the embedding), then at most one of the radial arcs A_i is contained in that disk.

   Solution: We view each arc as open, i.e., without its endpoints. Suppose w.l.g. that A_1 is contained in D. The endpoints of A_1 cuts C_1 into two arcs C' and C''. By the Jordan curve theorem and more precisely by theta's lemma, D is the union of two components D' and D'' bounded by $A_1 \cup C'$ and $A_1 \cup C''$ respectively. Since for each i the endpoints of A_i are radially opposite it must have one endpoint on C' and one on C''. Hence, if some A_i was contained in D it would intersect the boundary of D', hence A_1. This would contradict the hypothesis that K_n embeds in S_n.

6. Deduce from the previous question that in the embedding of G_n on S_n, if C_1 bounds a disk then this disk is a face.

   Solution: Let D be the disk bounded by C_1 and let $G_n \setminus C_1$ be the graph resulting from the removal from G_n of the vertices of C_1 and of their incident edges. Since $G_n \setminus C_1$ is connected it follows from the Jordan curve theorem applied to an open neighborhood of D that D contains either the whole of $G_n \setminus C_1$ or nothing. The first case is impossible by the previous exercise, so that D is indeed empty.

7. Show that S_2, and thus G_2, have genus at least 2. *Hint: If C_1 bounds a disk, use the previous questions. Otherwise, prove that $G_2 \setminus C_1$ is not planar, for example by finding a forbidden minor.*

   Solution: If C_1 bounds a disk, then S_2 has genus at least 2 according to Question 4. Otherwise, suppose by way of contradiction that G_2 embeds in the torus. Since C_1 does not bound a disk it cuts this torus into a cylinder. In particular, $G_2 \setminus C_1$ embeds into the cylinder, hence is planar. However, $G_2 \setminus C_1$ contains $C_2 \cup A_1 \cup A_2 \cup A_3$ as a subgraph, which is isomorphic to $K_{3,3}$. This would thus imply that $K_{3,3}$ is planar, and we have reached a contradiction. We conclude that S_2 has genus at least 2.

8. Show that S_n, and thus G_n, have genus at least n.

   Solution: We argue by induction on n. The base case $n = 2$ is the subject of the previous question. Suppose that for some $n \geq 2$, G_n has genus at least n and consider a cellular embedding of G_{n+1} in some orientable surface S_{n+1} (as we saw in Question 1). If C_1 bounds a disk in S_{n+1}, then, noting that K_{n+1} is a minor of G_{n+1}, we know from Question 4 that S_{n+1} has genus at least $n+1$. Otherwise, because $G_{n+1} \setminus C_1$ is connected and because the embedding is cellular, C_1 is non-separating in S_n. It follows that cutting S_{n+1} through C_1 decreases its genus by one. Now $G_{n+1} \setminus C_1$ embeds in $S_{n+1} \setminus C_1$ and contains G_n as a minor. By the induction hypothesis we conclude that $S_{n+1} \setminus C_1$ has genus at least n, implying that S_{n+1} has genus at least $n+1$.

The family of graphs G_n shows that one cannot obtain the inequality from question 1 in the other direction, i.e., bound the orientable genus by the non-orientable one.

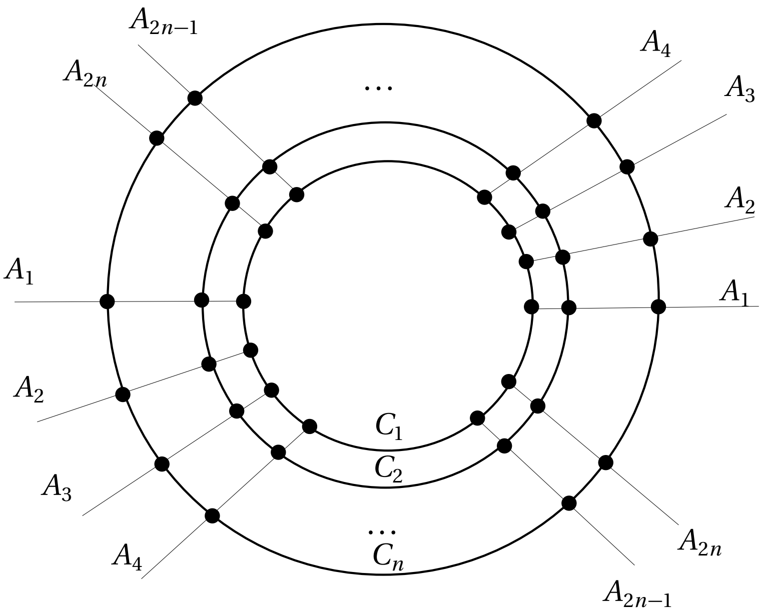


Figure 1: The family of graphs G_n.

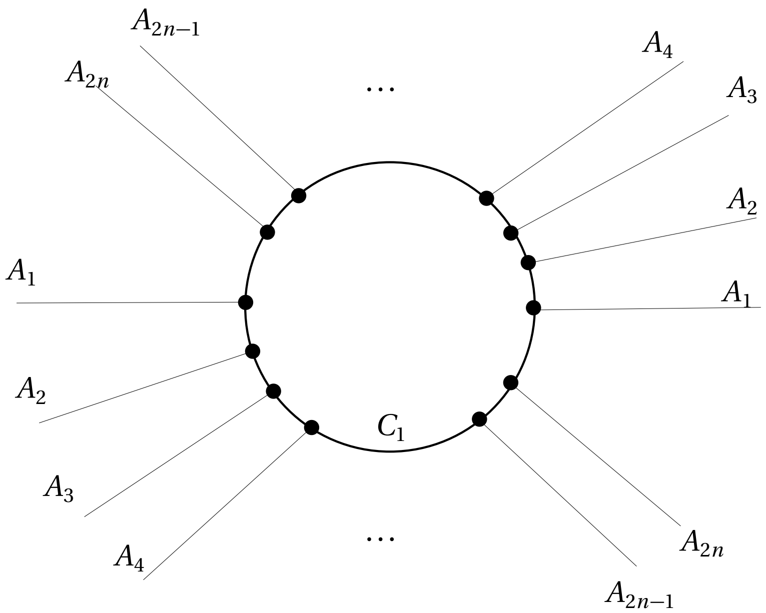


Figure 2: The family of graphs K_n.