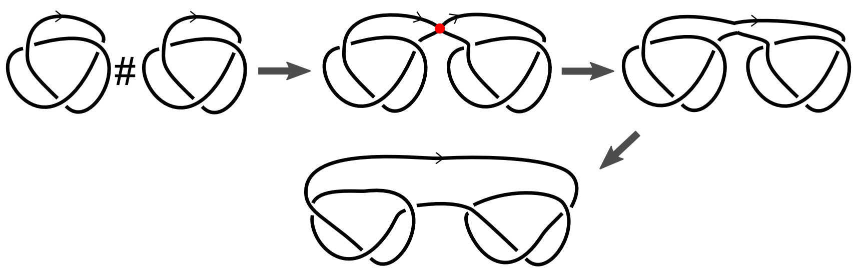
CR13: Computational Topology
Exercises #10

1. Recall that a **coloring** of a knot diagram is an assignment of each strand with one of three colors such that at least two colors are used, and at each crossing if two of the three incident strands have the same color then so has the third strand. Assuming that the three colors are fixed once for all, show that all the diagrams of a knot have the same number of colorings.

2. Compute the number of colorings of the trefoil knot.

3. The **connected sum** $K_1 \# K_2$ of oriented knots K_1 and K_2 is obtained by taking disjoint diagrams of the knots, deforming them so that they touch at one point (with opposite orientations) and performing a surgery operation at that point to connect the knots into a single one. The figure below illustrates the connected sum of two trefoil knots.



The connected sum is associative and commutative (think of one knot diagram as very small and connect it anywhere to another one. Clearly, you can slide the small knot along the other one to move the connecting point as you wish. In particular, if the other knot is already the sum of other knots you can permute the position of the small knot among the other ones.) What is the number of colorings of a connected sum of n trefoil knots?

4. Deduce from the preceding questions that there are an infinite number of non-equivalent knots.

5. A labelling of a knot diagram by a group G is an assignment of each strand with an element of G such that the assigned elements generate the whole group G, and at each crossing, denoting by $g, h, k \in G$ the assignment of the upper strand, the entering strand and the exiting strand respectively, we have $k = g^{-1}hg$ if the crossing is right-handed (the entering strand is to the right of the upper strand) and $k = ghg^{-1}$ otherwise. Show that all the diagrams of a given knot admit labellings by the same groups. In other words, the set of groups that can label a diagram is an invariant of the knot.

6. Show that the abelianization of a group (i.e., the quotient by the derived subgroup) that labels a knot must be cyclic (i.e., generated by a single element).

7. Show that the fundamental group π_1 of a knot is the most general group that labels its diagrams: for any group G such that the diagrams of the knot admit a labelling by G, there exists a surjective morphism $\pi_1 \twoheadrightarrow G$.