# CR13: Computational Topology
# Exercises #11

The objective of these exercises is to play with triangulations and 3-manifolds and their normal surfaces using the open source software Regina [1]. It is freely available at the address

`https://regina-normal.github.io/`

for Windows, Mac, Linux[1] and even for your iThings.

While we give some guidance through the software, getting lost is part of the fun. The user interface is very intuitive, including a “What’s this” tooltip (accessed with Shift-F1) and an exhaustive handbook (accessed with F1): use them extensively!

1. Create a new 3-manifold triangulation (the blue pyramid sign) and choose “Empty” as the Type of triangulation. Add a single tetrahedron using the “Add Tet” function. You can specify how to glue its faces by editing the empty field under “Face 012”, “Face 013”, etc. Glue the faces [0,1,2] and [1,3,0] by inputting 0(130) under the Face 012 (the 0 denotes the unique tetrahedron). This is the solid torus triangulation that we have already dealt with in Question 5 of Exercises #8 (or Exercise 3.6 in the lecture notes on Knots). Note that the software recognizes it instantly in the “Recognition” tab. In exercises #8, we computed its homology groups. Verify that it matches the computations in the Algebra tab.

2. Create a normal surface list (torus icon, leave all the default options) under this solid torus. This enumerates all the vertex normal surfaces in this triangulation. How many are there? The “Surface coordinates” tab outputs their normal coordinates. The “Link” column specifies when a given normal surface is the boundary of a small neighborhood of a vertex or an edge of the triangulation. Try to visualize the normal surfaces using their coordinates, and verify that the link description is correct.

3. Since our manifold is a solid torus, it is the complement of an unknot and should bound a vertex normal spanning disk by Lemma 4.6 of the lecture notes. Find it in the list.

4. Let us switch to a more interesting knot. The Figure 8 knot is pictured in Figure 1. It can be shown[2] that gluing two tetrahedra as in Figure 2 and removing a small neighborhood of the (unique) vertex gives a space homeomorphic to the complement of the Figure 8 knot. Such a triangulation of a knot complement is called an **ideal triangulation**. Input this triangulation $T$ into Regina, either by hand, or from the “Example triangulation” list. It should be automatically recognized as an ideal triangulation of the figure 8 knot complement (“Recognition” tab).

5. Create the list of vertex normal surfaces of that triangulation. What do you find?

   Since the Figure 8 knot is not the unknot, it does not bound a spanning disk. However, it still bounds surfaces of higher genus. The smallest possible genus of such a surface is called the **genus** of the knot. A proof similar to Lemma 4.6 in the lecture notes shows that such a surface of minimal genus can be found among the vertex normal surfaces of the knot complement. However, for that we require a real triangulation of the complement (where the boundary is realized by some faces) instead of an ideal one.

6. This transformation can be done automatically using the “Truncate ideal vertices” function in the “3-D Triangulation” menu[3]. Do it and observe that the number of tetrahedra has significantly blown up.

[1] On Ubuntu and a few other Linux distributions, you can simply install the regina-normal package.
[2] Visualizing how to go from Figure 1 to these two tetrahedra is an interesting, but pretty hard exercise.
[3] This might require you to delete the list of normal surfaces generated for the previous question.

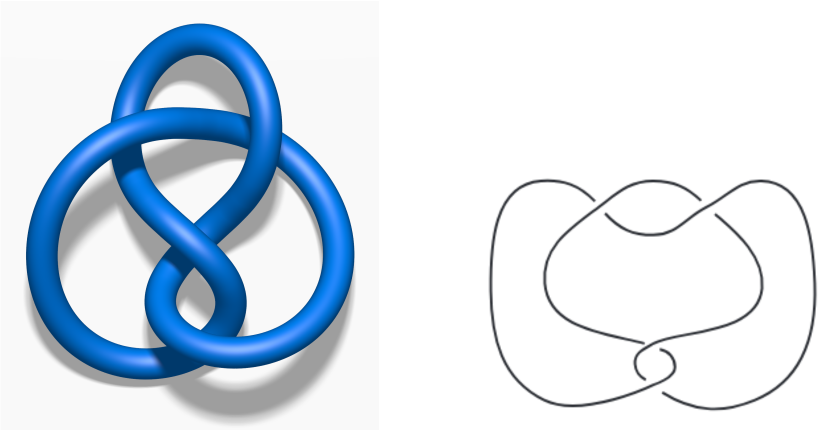

Figure 1: Two representations of the Figure eight knot.

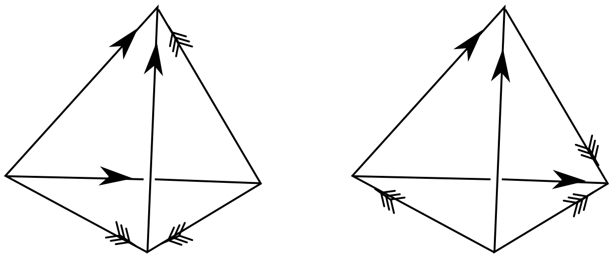

Figure 2: An ideal triangulation of the Figure 8 knot. There is a unique way to glue the faces of the first tetrahedron to the other so that the arrows match.

7. Unless you are working on a supercomputer, computing the list of vertex normal surfaces of that triangulation is prohibitive (remember that the algorithm is exponential). Try it! However, Regina has a "Simplify" function that locally simplifies triangulations, it is going to do wonders here. After such a simplification, compute the list of vertex normal surfaces.

   Out of all these vertex normal surfaces, we want to find one of minimal genus with a single boundary that is spanning, i.e., homotopically (or homologically) non-trivial in the boundary of $T$.

8. Can the normal surfaces that are edge links be spanning?

9. In the "Display coordinates" menu, choosing "Edge weights" describes normal surfaces by the number of times they intersect each edge of the triangulation. Edges on the boundary of the triangulation are denoted with a [B]. In our case, the boundary of $T$ is a triangulation of the torus with two triangles. How are the edge weights related to the homology of the boundary of a normal surface?

10. Use the previous question to deduce the genus of the Figure 8 knot.

# References

[1] Benjamin A. Burton, Ryan Budney, William Pettersson, et al. Regina: Software for low-dimensional topology. `http://regina-normal.github.io/`, 1999–2016.