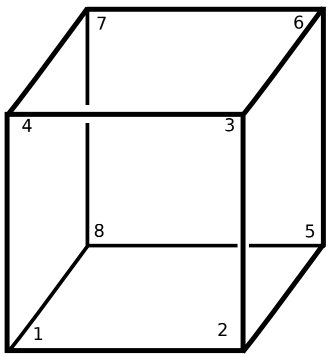
CR13: Computational Topology
Exercises #12
Due January 12

We saw that 3-manifolds could be obtained by gluing a collection of tetrahedra along their faces (see Exercise 3.6 in the lecture notes on Knots). More generally, given a set of three dimensional convex polyhedra $\mathscr{P} = \{P_1, \dots P_k\}$, a pairing of their faces, and a bijective correspondence between each pair of faces, one obtains a **quotient space** $\mathscr{P}/\!\sim$ by identifying each point in a face with the corresponding point in the paired face. We restrict to affine correspondences so that paired faces must have the same number of vertices and each bijective correspondence can be specified by the image of two consecutive vertices in a face. We forbid the pairing of a face with itself. Recall that a topological space is a closed (i.e. without boundary) 3-manifold if every point has a neighborhood homeomorphic to a 3-ball. For the quotient space $\mathscr{P}/\!\sim$ it is equivalent[1] to require that the **link** of every point, i.e., the set of points at a fixed small distance[2] of the point is a 2-sphere. Indeed, each point has a neighborhood which is a cone over its link, hence a 3-ball if the link is a 2-sphere. The **Euler characteristic** of a quotient space is the alternated sum $\chi(\mathscr{P}/\!\sim) = v - e + f - k$ where v, e, f, k are respectively the number of vertices, edges, faces and polyhedra of $\mathscr{P}/\!\sim$. It can be shown that two homeomorphic quotient spaces have the same characteristic.

Let $C = [0,1]^3$ be the standard cube. Consider the case where $\mathscr{C} = \{C\}$ and each face is identified with its opposite face. The correspondences are given by a translation followed by a clockwise quarter turn for each of the two pairs of vertical faces and by a translation followed by a counter-clockwise quarter turn for the third (horizontal) pair. Hence, in the following drawing the face 1234 is identified with the face 5678 with the idenfications of the vertices in this order.



1. Compute the number of vertices, edges, faces and cells of $\mathscr{C}/\!\sim$.

2. Describe the link of the vertices of $\mathscr{C}/\!\sim$ and conclude that $\mathscr{C}/\!\sim$ is not a 3-manifold.

3. Suppose that every polyhedron in $\mathscr{P}$ is a tetrahedron. Let v be a vertex of a quotient space $\mathscr{P}/\!\sim$ and let e_v, f_v, t_v the respective number of edges, faces and tetrahedra of $\mathscr{P}/\!\sim$ incident with v counting multiplicity. (Hence, a loop edge of $\mathscr{P}/\!\sim$ counts twice for its identified endpoints and similarly for higher dimensional cells.) Express the Euler characteristic $\chi(L_v)$ of the link L_v of v in terms of e_v, f_v and t_v.

[1] In dimension higher than four this might not be equivalent. There are quotient spaces of polyhedra that form a manifold but whose vertex links are not spheres. An example is provided by the double suspension on a homology sphere.

[2] Here, we consider the quotient distance assuming that the correspondences are isometric.

4. Compute the Euler characteristic of $\mathscr{P}/\sim$ in terms of the sum $\sum_v(\chi(L_v)-2)$, where the sum runs over all the vertices of $\mathscr{P}/\sim$.

5. Conclude that $\mathscr{P}/\sim$ is a closed 3-manifold if and only if its Euler characteristic is zero.

6. Apply the preceding question to show directly that the above quotient space $\mathscr{C}/\sim$ is not a 3-manifold.

7. Let D be the regular dodecahedron. Consider the case where $\mathscr{D}=\{D\}$ and each pentagonal face is identified with its opposite face. The correspondences are given by a translation followed by a clockwise turn of $\pi/5$ (1/10 of a complete turn) for each pair of faces. Show that $\mathscr{D}/\sim$ is a manifold.

8. Compute a presentation of the fundamental group of $\mathscr{D}/\sim$. You may use the fact that the fundamental group of a cellular complex (an assembly of vertices, edges, faces,...) only depends on its 2-skeleton, i.e. the subcomplex induced by the vertices, edges and faces. Now, you may "invert" the construction of the two dimensional complex associated to a group presentation as we saw in the lecture on undecidability in topology.

9. The Hurewicz theorem states that the first homology group with *integer coefficients* of a cellular complex is isomorphic to the abelianization of its fundamental group. Deduce that $\mathscr{D}/\sim$ has the same homology as a 3-sphere (again with integer coefficients).

It can be shown that the fundamental group of $\mathscr{D}/\sim$ is nontrivial (by surjecting it onto the group of orientation preserving symmetries of the dodecahedron). It follows that $\mathscr{D}/\sim$ is not homeomorphic to a 3-sphere, although it possesses the same homology. For this reason it is called the **Poincaré homology sphere**. Indeed, Poincaré, who constructed this space, first thought that it was homeomorphic to a sphere since it had the same homology. He later realized that this was not the case by computing the fundamental group and formulated his famous conjecture that a simply connected (i.e., with trivial fundamental group) compact 3-manifold is necessarily homeomorphic to a 3-sphere. The conjecture was eventually proved correct by Grigori Perelman in 2003. It was suggested by Jean-Pierre Luminet in 2003 that the shape of the universe is a Poincaré homology sphere.