CR13: Computational Topology
Exercises #4
Due October 19th

1. Let G be a connected graph that is not a tree. Recall that the orientable (resp. non-orientable) genus $g(G)$ (resp. $\tilde{g}(G)$) of a graph is the smallest genus of an orientable (resp. non-orientable) surface on which it embeds. Show that they satisfy:

$$\tilde{g}(G) \leq 2g(G) + 1.$$

Let G_n be the family of graphs in Figure 1. The A_i are edges that wrap around and identify opposite points.

2. Show that for every n, G_n embeds in the projective plane.

3. Show that for every n, G_n embeds on the orientable surface of genus n.

   Henceforth, we assume that G_n is embedded on an orientable surface S_n of genus g. The subgraph K_n is defined in Figure 2, and inherits an embedding on S_n from the embedding of G_n.

4. Show that if C_1 bounds a face that is a disk, then S has genus at least n. *Hint: Compute the faces of K_n.*

5. Show that if C_1 bounds a disk D (but not necessarily a face of the embedding), then at most one of the radial arcs A_i is contained in that disk.

6. Deduce from the previous question that in the embedding of G_n on S_n, if C_1 bounds a disk then this disk is a face.

7. Show that S_2, and thus G_2, have genus at least 2. *Hint: If C_1 bounds a disk, use the previous questions. Otherwise, prove that $G_2 \setminus C_1$ is not planar, for example by finding a forbidden minor.*

8. Show that S_n, and thus G_n, have genus at least n.

The family of graphs G_n shows that one cannot obtain the inequality from question 1 in the other direction, i.e., bound the orientable genus by the non-orientable one.

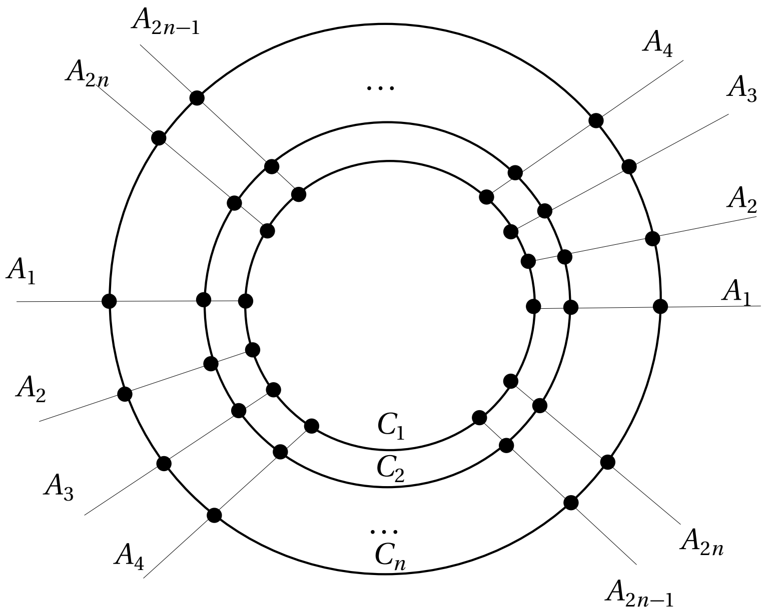

Figure 1: The family of graphs G_n.

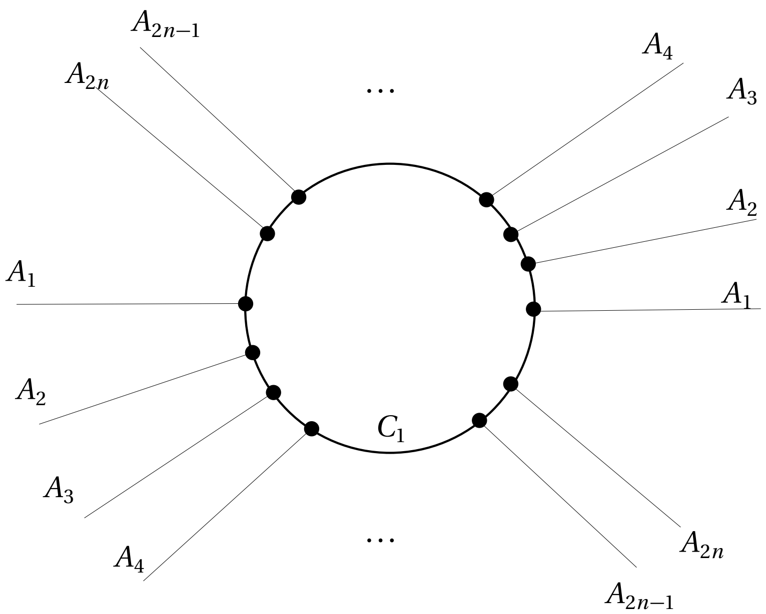

Figure 2: The family of graphs K_n.