CR13: Computational Topology
Exercises #6

A *covering* of a graph G is a projection of another graph H onto G such that H looks locally as its image in G. For a formal definition it appears more convenient to define an edge of a graph as a pair of opposite **arcs**, where the opposite of an arc a is denoted by a^{-1}. Hence, a graph $G = (V, A)$ is defined by a set of vertices V, a set of pairs of opposite arcs A and a map $o : A \to V$ that assigns to every arc its origin vertex. A loop edge is thus a pair of opposite arcs with the same origin. The **star** of a vertex v, denoted by $\mathrm{Star}(v)$, is the set of arcs with origin v. A map $p : A_H \to A_G$ between the arc sets of two graphs H and G defines a **covering** if

- p sends opposite arcs to opposite arcs,
- p is onto,
- p sends stars to stars, hence defines a map between the vertex sets (assuming H is connected),
- for every vertex w of H, the restriction $p : \mathrm{Star}(w) \to \mathrm{Star}(p(w))$ is bijective.

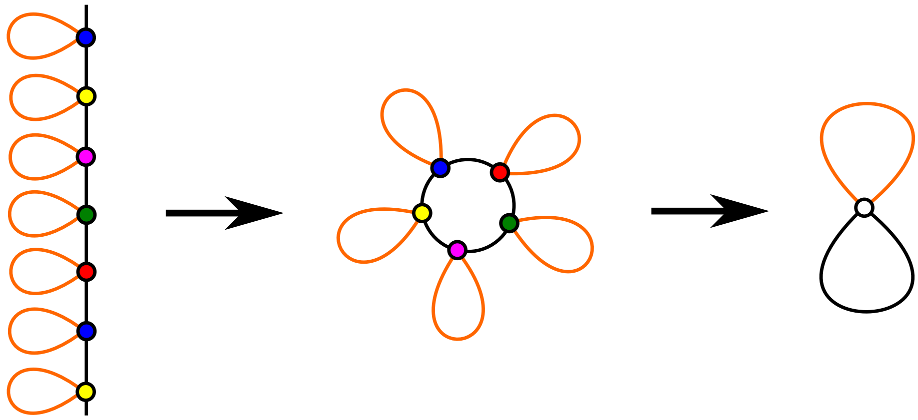
Let $p : (H, w) \to (G, v)$ be a covering where w is a vertex of H that projects to v. Recall that two loops are homotopic if they can be related by a sequence of insertions or removals of **spurs**, i.e., subpaths of the form (a, a^{-1}).

1. Show the *unique lift property*, that is for any path ℓ with origin v in G there is a unique path c with origin w in H such that $p(c) = \ell$. Path c is called a **lift** of ℓ (from w).

2. Show that two loops with basepoint v are homotopic if and only if their lifts are homotopic (as paths with fixed endpoints).

3. Show that p induces a group morphism $p_* : \pi_1(H, w) \to \pi_1(G, v)$ and that this morphism is one-to-one (injective). How can you tell from the lift of a loop in G whether it belongs to $p_*(\pi_1(H, w))$?

4. Deduce from the preceding question and the figure below that the free group over 2 elements contains as a subgroup the free group over k elements for any $k \in \mathbb{N}$ or even over a countably infinite set of elements!



5. Let G be a *bouquet of circles*, that is a connected graph with a single vertex v. Hence, each arc a determines a loop with homotopy class $[a]$. Given a subgroup U of $\pi_1(G,v)$, we define a graph G_U by

   - $V(G_U) = \{v\} \times \{Ug\}_{g\in\pi_1(G,v)}$,
   - $A(G_U) = A(G) \times \{Ug\}_{g\in\pi_1(G,v)}$,
   - $o(a,Ug) = (v,Ug)$ and $(a,Ug)^{-1} = (a^{-1}, Ug[a])$,

   where Ug denotes the right coset representative of g with respect to U in $\pi_1(G,v)$. Schematically, the typical edge of G_U is

   $$(v,Ug)\bullet \underset{(a^{-1},Ug[a])}{\overset{(a,Ug)}{\rightleftarrows}} \bullet(v,Ug[a])$$

   and let $p_U : A(G_U) \to A(G)$ be the projection on the first component.

   5a. Show that p_u is a covering map.

   5b. Let $\lambda = (a_1,\dots,a_k)$ be a loop based at v in G. Setting $w = (v,U)$, express the target vertex of the lift of λ from w in G_U in terms of $[a_1],\dots,[a_k]$.

   5c. Deduce that $\pi_1(G_U,w)$ is isomorphic to U.

6. Deduce from the preceding exercise that every subgroup of a free group is itself a free group. This important result in combinatorial group theory is known as the **Nielsen-Schreier theorem**.