CR13: Computational Topology
Exercises #7

Recall that the *cycle space* $Z(G,\mathbb{Z}_2)$ of a graph $G=(V,E)$ is the subspace of chains, *i.e.* of formal linear combinations of edges with[1] $\mathbb{Z}_2$ coefficients, with zero boundary. As usual, the boundary operator is the linear extension of the map that sends an edge uv to the formal difference $v-u$ of its endpoints. If we replace the $\mathbb{Z}_2$ coefficients by any ring R of coefficients we obtain the cycle space $Z(G,R)$ over R. In general, when $1 \neq -1$ in R, one should consider a chain as a linear combination of *oriented* edges, identifying the opposite $-\vec{e}$ of an oriented edge $\vec{e}$ with the oppositely oriented edge. Equivalently, we may assume that every unoriented edge is given a default orientation and consider chains as linear combinations of *unoriented* edges using the default orientation to define the boundary operator (so that for an edge with endpoints u and v we have $\delta_1 e = v-u$ if the default orientation goes from u to v and $\delta_1 e = u-v$ otherwise).

Given a weight function $w: E \to \mathbb{R}_+$, we define the weight of a chain $c=\sum_{e\in E}\alpha_e e$ as $w(c)=\sum_{e\in E}|\alpha_e|w(e)$. This is well defined if the coefficients are for instance in $\mathbb{Z}_2$ or $\mathbb{Z}$ with the obvious interpretation of $|\cdot|$. A minimum cycle basis is then a basis of the cycle space with minimum total weight.

Let G be the *generalized Petersen graph* obtained by taking two copies of the cycle C_{11} with respective vertex sets $\{I_0,I_1,\dots,I_{10}\}$ and $\{O_0,O_1,\dots,O_{10}\}$, adding the 11 edges $O_j I_{3j}$ between the two cycles, with $0\le j\le 10$ and taking indices modulo 11. The cycles are respectively called the *inner* cycle and the *outer* cycle and the other edges are called *spokes*. We consider the weight function :

$$w(e)=\begin{cases}5 & \text{if } e \text{ is an inner edge,}\\ 4 & \text{if } e \text{ is an outer edge,}\\ 12 & \text{if } e \text{ is a spoke.}\end{cases}$$

Our aim is to compare the minimum bases for $Z(G,\mathbb{Z}_2)$ and for $Z(G,\mathbb{Z})$.

1. Compute all the possible weights of a simple cycle in G that uses at most two spokes.

2. Let $B_{\mathbb{Z}_2}$ be the set of cycles composed of the outer cycle and the 11 cycles formed with one outer edge, two spokes and 3 inner edges. Show that B is the unique minimum basis for $Z(G,\mathbb{Z}_2)$. Hint: Notice that every cycle in G uses an even number of spokes.

3. Let T be a spanning tree of G. For every chord of T consider the *fundamental cycle* obtained by joining the chord endpoints with a simple path in T. Show that the set of fundamental cycles (with a chosen orientation) is a basis of $Z(G,\mathbb{Z})$. We call it the *fundamental basis* associated to T. Given any cycle in $Z(G,\mathbb{Z})$, how can you read the coefficients of its decomposition in the fundamental basis?

[1] For convenience, we denote by $\mathbb{Z}_2$ the field $\mathbb{Z}/2\mathbb{Z}$.

4. Show that $B_{\mathbb{Z}_2}$ is not a basis of $Z(G,\mathbb{Z})$. You may decompose $B_{\mathbb{Z}_2}$ over the fundamental basis B_T associated with the spanning tree T composed of all the spokes and all the inner edges but one. Alternately, you may compute the sum of the cycles in $B_{\mathbb{Z}_2}$ (with coherent orientations of the cycles).

5. Consider the set of cycles $B_{\mathbb{Z}}$ obtained from $B_{\mathbb{Z}_2}$ by replacing the outer cycle with the simple cycle composed of an inner edge, two spokes and four outer edges. Compute the weight of $B_{\mathbb{Z}}$ and show that $B_{\mathbb{Z}}$ is a minimum basis for $Z(G,\mathbb{Z})$.