CR13: Computational Topology
Exercises #9
Due November 30

Throughout the homework, when we use *homology*, we mean homology on surfaces with coefficients in $\mathbb{Z}$. Note that this differs from the course on surfaces where we use $\mathbb{Z}_2$ coefficients, but the lecture notes on homology introduces the (very similar) formalism.

We denote by S_g an orientable surface of genus g, which, as usual, is endowed with a cellularly embedded graph G. By a closed curve on S_g, we mean a map $\mathbb{S}^1 \to S_g$. A closed curve is *simple* if that map is injective. A simple closed curve γ is *nonseparating* if $S_g \setminus \gamma$ has a single connected component. A closed curve γ *induces* a homology class $[\gamma]$ by "pushing" it into a closed walk on the graph G, where, as a multiset of oriented edges, it corresponds to a homology class. This "pushing" can be done in multiple ways, but since the resulting walks differ by boundaries of faces, they correspond to the same homology class. A homology class c is *represented* by a closed curve γ if $c = [\gamma]$. Two closed curves are *homologous* if they induce the same homology class.

Recall from Exercise sheet #5 that an element $(p, q) \in \pi_1(T) = H_1(T) = \mathbb{Z}^2$ can be represented by a simple closed curve if and only if p and q are relatively prime. In this homework, you can use this fact without reproving it. The goal of this homework is to obtain a similar result for other surfaces.

An element v in $H_1(S_g)$ is *primitive* if $v \neq nw$ for any $w \in H_1(S_g)$ and any integer $n \geq 2$. We denote the greatest common denominator of two integers a and b by $gcd(a, b)$.

1. Prove that if a nonzero element of $H_1(S_g)$ can be represented by a simple closed curve, then it is primitive.

2. Prove that if $g = 1$, if an element is primitive then it can be represented with a simple closed curve.

3. Prove that S_g admits a basis B for $H_1(S_g)$ represented by simple closed curves $a_1, b_1, \dots a_g, b_g$ such that a_i intersects b_i exactly once, and there are no other intersections between these curves. We denote by N_i a small neighborhood of the pair of curves a_i and b_i.

4. Let v be an element of $H_1(S_g)$, of which the decomposition on the basis B is denoted $(v_1, w_1, \dots, v_g, w_g)$. Show that for each i, there is a nonseparating simple closed curve γ_i in N_i so that
$$gcd(v_i, w_i)[\gamma_i] = v_i[a_i] + w_i[b_i].$$

5. Show that if there exist two nonseparating disjoint and non-homologous simple closed curves α and β on S_g such that $[\alpha] + [\beta] \neq 0$, then there also exists a simple nonseparating closed curve representing $[\alpha] + [\beta]$.

6. Show that there exists a simple closed curve $\gamma_{1,2}$ such that $gcd(v_1, w_1, v_2, w_2)[\gamma_{1,2}] = v_1[a_1] + w_1[b_1] + v_2[a_2] + w_2[b_2]$. *Hint: Use the previous question and the Euclidean algorithm.*

7. By induction, show that if an element of $H_1(S_g)$ is primitive, then it can be represented with a simple closed curve.

8. Given a closed curve γ on a surface, provide an algorithm to determine whether γ is homologous to a simple closed curve γ'. What is the complexity of your algorithm?