Algorithmic Topology and Groups
Exercises #2

1. Recall the definition from the lecture notes:

   If $E' = B \cup \iota(B)$ is a subset of edges of a graph $G = (V, A, o, \iota)$, the contraction of E' in G is the graph $G/E' = (V', A', o', \iota')$ where $A' = A \setminus (B \cup \iota(B))$ and $V' = V/\approx$, where $\approx$ is the transitive and reflexive closure of the relation $\bigcup_{b \in B}(o(b), o \circ \iota(b))$, and o' and ι' are defined in the obvious way.

   There is an obvious morphism $c_H : G \to G/H$ sending each vertex to its class modulo $\approx$ and each edge in H to the class of its origin vertex. When H is finite, c_H is a composition of edge contractions. When H is infinite, show that G/H is a direct limit of appropriate morphisms.

2. Verify that the Euler characteristic of finite graphs, i.e., the number of vertices minus the number of edges, is an invariant with respect to combinatorial equivalence. Viewing the set of integers as category with one arrow $m \to n$ whenever $m \le n$, is this invariant a functor?

3. Denote by G^2 the subgroup generated by the squares of the elements of a group G. Show that G^2 is normal and that G/G^2 is abelian. Can you directly prove that $[G, G] \subseteq G^2$?

4. Let $w = x_{i_1}^{n_1} x_{i_2}^{n_2} \cdots x_{i_k}^{n_k}$ be a product in the generators $X = \{x_i\}_{i \in I}$ of the free group $F(X)$. Show that $w \in [F(X), F(X)]$ if and only if, for every $i \in I$, the exponents of the occurrences of x_i in w sums up to zero.

5. Show that the one dimensional homology of a graph G is the direct sum of the one dimensional homology of its 2-connected components (the blocks of G). (Hint: consider the map sending a cycle to its traces over the 2-connected components of the graph.)

6. Let T be a spanning tree of a connected graph G. Denote by C the set of chords of T, and for any chord $a \in C$ denote by $T[a]$ the simple cycle intersecting C in a. Show that for a field Γ, $H_1(G, \Gamma)$ is a vector space of dimension $\beta_1(G)$ (in general a direct sum of copies of Γ) and that $B = \{T[a]\}_{a \in C}$ is a basis of $H_1(G, \Gamma)$.

7. Let G be a finite connected graph with a positive weight function w on its edges. Show that a basis B of $H_1(G, \mathbb{Z}/2\mathbb{Z})$ has minimum weight $\sum_{b \in B} w(b)$ if and only if the list $\ell(B)$ of lengths of the cycles of B written in increasing order is minimal for the lexicographic order.

8. Let m, n be two coprime positive integers. The **Petersen** graph $G(n, m)$ is obtained from two cycles of length n, $C_1(n)$ and $C_2(n)$, with vertices indexed circularly by $\mathbb{Z}/n\mathbb{Z}$, by connecting vertex i of $C_1(n)$ with vertex mi of $C_2(n)$. The

Petersen graph is thus a cubic graph with $2n$ vertices and $3n$ edges. The connecting edges between the two cycles are called *spokes*.

Consider the Petersen graph $G(11,4)$ with the following edge weights. Each edge of $C_1(11)$ has weight 5, each edge of $C_2(11)$ has weight 4, and each spokes has weight 12.

- Describe the set Σ_k of simple cycles of $G(11,4)$ with minimum weight that contain exactly k spokes for $k = 0,1,2,3$. Compute this weight for $k = 0$ and $k = 2$.
- Argue that the cycles in $B := \Sigma_0 \cup \Sigma_2$ is the unique minimum weight $\mathbb{Z}/2\mathbb{Z}$-homology basis.
- Argue that B is not a $\mathbb{Z}$-homology basis.
- Let b be a simple cycle of minimum weight that contains exactly one edge of C_1. Argue that $B' = \Sigma_2 \cup \{b\}$ is a minimum weight $\mathbb{Z}$-homology basis.
- Can you find a graph with uniform weights for which none of the minimum $\mathbb{Z}/2\mathbb{Z}$-homology bases is a $\mathbb{Z}$-homology basis?