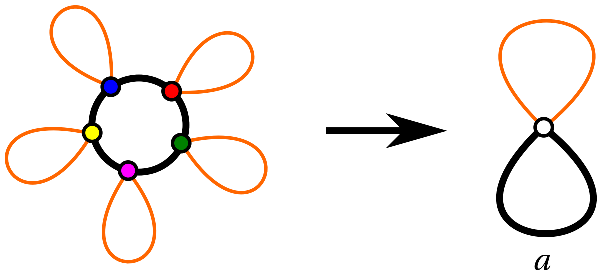
Algorithmic Topology and Groups
Exercises #3

1. Let $p : \text{Flower}_5 \to B_2$ be the covering represented in the following Figure.

   Call a the lower loop edge of B_2. On what condition related to a does a loop of B_2 lift to a loop in Flower_5? Deduce that for any vertex v of Flower_5, we have $p_*\pi_1(\text{Flower}_n, v) \lhd \pi_1(B_2)$, i.e. $\pi_1(\text{Flower}_n, v)$ is normal in $\pi_1(B_2)$. What is the quotient group? Justify your answer in two different ways.

2. Let f be a morphism from the covering $p : H \to G$ to the covering $q : K \to G$. Consider a vertex v in H and a path α in G with initial vertex $p(v)$. Show the identity
   $$f(v).\alpha = f(v.\alpha)$$

3. Show that a covering morphism $$\begin{array}{ccc} (H,v) & \longrightarrow & (H,v) \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p} & \\ & (G,u) & \end{array}$$ with H connected, must be the identity.

4. Consider the Petersen graph $P = P(5,2)$ (see Exercise sheet #2). Let r be the autmorphism of P acting on $C_1(5)$ by a one-shift (vertex i is sent to vertex $i+1$). What is the quotient graph $P/\langle r\rangle$? Can you obtain the same quotient graph as a quotient of the graph (1-skeleton) of the 3 dimensional cube?

5. Give a characterisation of the embedding of $\pi_1(P)$ induced by the quotient in the previous exercise.

6. Prove that there exists a free action of $\mathbb{Z}/n\mathbb{Z}$ on the complete graph K_n if and only if n is odd. What is the quotient graph for $n = 5$? Give a characterisation of the corresponding embedding of $\pi_1(K_5)$ in the fundamental group of the quotient.

7. Give an example of a graph morphism between connected graphs whose edge or vertex fibers all have the same size though the morphism is not a covering.