Algorithmic Topology and Groups
Exercises #6

1. Let $G = \langle S \mid R\rangle$ be a group presentation and let r be a word in S that is a consequence of R, *i.e.* such that $r =_G 1$. Show that r is freely equivalent (*i.e.*, inserting or removing ss^{-1} or $s^{-1}s$ factors) to a word of the form
$$\prod_{j=1}^{k} g_j r_j^{\varepsilon_j} g_j^{-1},$$
where the g_j are words over S, $r_j \in R$, and $\varepsilon_j \in \{-1, 1\}$.

2. Recall that a Tietze transformation on a group presentation $G = \langle S \mid R\rangle$ may either add a relation which is a consequence of R or add a new generator s with a new relation sw, where w is any word over S. Show that the Tietze transformations indeed produce isomorphic groups.

3. Let $G = \langle X \mid R\rangle$, $H = \langle Y \mid T\rangle$ be group presentations and let $\varphi : A \hookrightarrow G$ and $\psi : A \hookrightarrow H$ be group embeddings. The **free product with amalgamation** of G and H with respect to φ and ψ is the pushout of the diagram $G \hookleftarrow A \hookrightarrow H$. It is denoted by $G \star_{\varphi=\psi} H$.

   Show that for any set S of generators of A, $G \star_{\varphi=\psi} H$ has presentation
$$\langle X \cup Y \mid R \cup T \cup \{\varphi(s)(\psi(s))^{-1}\}_{s\in S}\rangle$$

4. **Seifert–van Kampen theorem for surfaces.** Let M_1, M_2 be connected finite maps with boundary (*i.e.* having perforated faces). Suppose that B_1, B_2 are perforated faces of M_1, M_2, respectively, with simple and isomorphic facial circuits ∂B_i, $i = 1, 2$. Let $v_1 \in \partial B_1$ and $v_2 \in \partial B_2$ be corresponding vertices under this isomorphism. Consider the map $M = M_1 + M_2$ obtained by identifying ∂B_1 and ∂B_2 according to this isomorphism. Also consider the group morphisms $\varphi_i : \mathbb{Z} \to \pi_1(M_i, v_i), 1 \mapsto [\partial B_i]$. Show that
$$\pi_1(M, v) \simeq \pi_1(M_1, v_1) \star_{\varphi_1=\varphi_2} \pi_1(M_1, v_2)$$
where v results from the identification of v_1 and v_2. (Hint: you may start with adapted presentations of $\pi_1(M_i, v_i)$.)

5. Show that the groups $G = \langle x, y \mid x^3, y^2, (xy)^2\rangle$ and $G' = \langle y, z \mid (zy)^3, y^2, z^2\rangle$ are isomorphic. Can you recognize that group?

6. Let G be the group $\langle x, y \mid x^5, y^3, yxy^{-1}x^{-3}\rangle$. How many elements does G have? *Hint: It's not 15.*