# Homology

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## Contents



As we have seen in the previous lecture, as the dimension of the space under study increases, most topological problems become very quickly undecidable. In this lecture, we investigate one of the only topological notions that remains computable no matter the dimension: homology.

# 1 Complexes

In order to talk about algorithms, we first explain how to describe and manipulate topological spaces of arbitrary dimension. Following the path that we used for surfaces, we will build complicated spaces by gluing together fundamental blocks, called simplices. The resulting object is generally called a complex. However, there are many different ways in which this can be done, each of which having advantages and disadvantages. We will focus on two closely related concepts: we first introduce simplicial complexes, which are very simple to define but sometimes cumbersome to use, which is why we generalize them slightly to $\Delta$-complexes.

An **affine simplex** in dimension $n$ is the convex hull of $n+1$ affinely independent points in $\mathbb{R}^p$ for some big enough $p$ and a **simplex** is the topological space defined by this affine simplex. The points are called the **vertices** of the simplex and a **face** of a simplex $\sigma$ is a simplex defined by a subset of the vertices of $\sigma$. A **simplicial complex** is a collection $X$ of simplices such that every face of a simplex in $X$ is also in $X$, and any two simplices of $X$ intersect in a common face, possibly empty. The **dimension** of a simplicial complex is the maximal dimension of its simplices.

For technical reasons, we will require *orientations* on simplices. For a simplex $\sigma$ with vertices $s_0, \dots, s_n$, we consider two permutations on the vertices $s_i$ to be equivalent if they have the same parity, and an orientation is such an equivalence class. We will use the notation $[s_0, \dots, s_n]$ to denote a simplex endowed with the orientation induced by the permutation $(s_0, \dots, s_n)$. For an oriented simplex $\sigma$, we denote by $-\sigma$ the same simplex with the opposite orientation. In the rest of these notes, we will always consider that the simplices in a simplicial complex are oriented in an arbitrary fashion.

The notion of simplicial complex is a bit awkward, because it requires some ambient space, and more restrictive than the notions of triangulations that we saw for surfaces and 3-manifolds: it allows gluings between different simplices, but not identifications within a single simplex: for example the weird triangulation of Exercise 3.6 in the lecture on knots and 3-manifolds is not a simplicial complex. The notion of **$\Delta$-complex** is a generalization of simplicial complexes that allows us to identify the faces of a collection of simplices pretty much as we want. An $n$-simplex is the image of an affine simplex by a homeomorphism. A $\Delta$-complex is the last space in an inductively defined sequence of topological spaces $X^{(0)} \subseteq \dots \subseteq X^{(n)} = X$, where each space $X^{(k)}$ is called the $k$-skeleton of $X$. For each integer $k > 0$, we inductively construct the $k$-skeleton $X^{(k)}$ by attaching a set of $k$-simplices to the $(k-1)$-skeleton $X^{(k-1)}$. Each $k$-simplex $\Delta_k$ is attached by a **gluing map** $\sigma : \partial\Delta_k \to X^{(k-1)}$ that maps the interior of each face of $\Delta_k$ homeomorphically to the interior of a simplex in $X^{(k-1)}$ of the same dimension. Such maps are called **cellular**.

For example, a triangulated map is a 2-dimensional $\Delta$-complex. If the underlying graph is a simple graph this can even be realized as a simplicial complex. Note however that some maps that we used for surfaces (for example the polygonal schemes) are complexes which are still not $\Delta$-complexes. They fit within a further generalization which is the notion of **polyhedral complexes**, which allows the building blocks to be arbitrary polyhedra and not just simplices. We will not define it in this course, but everything homological works the same for them. An even further generalization leads to the notion of a **CW-complex**, where the gluing maps are not required to be cellular. We will also not define these, and just say that everything homological works almost the same, except that some care must be taken when handling attaching maps.

# 2 Homology

## 2.1 Chain complexes

The homology of a $\Delta$-complex is obtained by first defining a *chain complex* out of a $\Delta$-complex, and then taking the homology of this chain complex. This chain complex depends on the choice of a *coefficient ring* which we will denote by $R$ in these notes. The correct level of generality to study homology will involves modules over this ring $R$, but for the reader not very acquainted with commutative algebra, only considering the cases where $R$ is the set of integers $\mathbb{Z}$ (leading to finitely generated abelian groups) or the $p$-element field $\mathbb{Z}_p$ for $p$ prime (leading to vector spaces over $\mathbb{Z}_p$) is enough for intuition and for most practical purposes.

The space of **k-chains** of a $\Delta$-complex $X$ is the set of formal linear combinations over $R$ of its oriented simplices of dimension $k$, i.e., a k-chain is a function $\alpha : X_k \to R$ where $X_k$ is the set of simplices of dimension $k$ in $X$. A $k$-chain is generally written as

a formal sum $\sum_i \alpha_i \Delta_{k,i}$ where $\Delta_{k,i}$ is the $i$th oriented $k$-simplex in $X$ and $\alpha_i = \alpha(\Delta_{k,i})$. Morally, $\alpha$ describes how many times one picks each simplex in the complex $X$. We denote by $C_k(X)$ the space of k-chains of a complex $X$. It is isomorphic (as a module) to $R^{n_k}$ where $n_k$ denotes the number of $k$-simplices in $X$.

The **boundary** of an oriented simplex is defined by

$$\partial_k [s_0, \ldots, s_k] = \sum_{i=0}^{n} (-1)^i [s_0, \ldots, \hat{s_i}, \ldots, s_k]$$

where $\hat{s_i}$ denotes the omission of the vertex $s_i$. Thus the boundary of a $k$-chain is a $(k-1)$-chain, and by linear extension the boundary operator $\partial_k$ can be defined on the set of $k$-chains $C_k(X)$ in the following way:

$$\begin{array}{rcl} \partial_k : C_k(X) & \to & C_{k-1}(X) \\ \sum_i \alpha_i \Delta_{k,i} & \mapsto & \sum_i \alpha_i \partial_k(\Delta_{k,i}). \end{array}$$

The key relation in homology is that the boundary of a boundary is empty:

**Lemma 2.1.** $\partial_{k-1} \circ \partial_k = 0$.

PROOF. By linearity, it is enough to prove that the boundary of the boundary of a $k$-simplex is the empty $(k-2)$-chain. This is a matter of computation:

$$\begin{array}{rl} \partial_{k-1}[s_0, \ldots, \hat{s_i}, \ldots, s_k] & = \\ \sum_{j<i} (-1)^j [s_0, \ldots, \hat{s_j}, \ldots, \hat{s_i}, \ldots s_k] - \sum_{j>i} (-1)^j [s_0, \ldots, \hat{s_i}, \ldots, \hat{s_j}, \ldots s_k] & \end{array}$$

and thus

$$\begin{array}{rl} \partial_{k-1} \circ \partial_k [s_0, \ldots, s_n] & = \\ \sum_{i=0}^{k} \sum_{j<i} (-1)^{i+j} [s_0, \ldots, \hat{s_j}, \ldots, \hat{s_i}, \ldots s_k] - \sum_{i=0}^{k} \sum_{j>i} (-1)^{i+j} [s_0, \ldots, \hat{s_i}, \ldots, \hat{s_j}, \ldots s_k] & = 0 \end{array}$$

since both sums are equal. □

Thus, for an $n$-dimensional $\Delta$-complex $X$, we have a sequence of boundary morphisms linking the chain groups:

$$0 \to C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \xrightarrow{\partial_{n-1}} \ldots \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \to 0,$$

where $\partial_{k-1} \circ \partial_k = 0$ and 0 denotes the trivial group. Such a sequence is called a **chain complex**.

## 2.2 Simplicial homology

The space of $k$-**cycles**, denoted by $Z_k(X)$, is the space of simplices without boundary, i.e., the kernel of the morphism $\partial_k$. The space of $k$-**boundaries**, denoted by $B_k(X)$ is the image of $\partial_{k+1}$. Since $\partial_{k-1} \circ \partial_k = 0$, we have $B_k(X) \subseteq Z_k(X)$, and this allows us to define the $k$**th homology group** $H_k(X)$ as the quotient

$$H_k(X) = Z_k(X)/B_k(X) = \ker \partial_k / \operatorname{Im} \partial_{k+1}.$$

The collection of all the homology groups $H_k(X)$ is usually denoted by $H_*(X)$.

Obviously, this definition depends heavily on the $\Delta$-complex we consider, but it turns out that the homology groups are invariant under homeomorphism.

**Theorem 2.2.** *If $X$ and $Y$ are homeomorphic $\Delta$-complexes, then $H_*(X) = H_*(Y)$.*

One naive way to establish it could be to follow the same line of thought as when we proved the invariance of many topological properties for surfaces:

PROOF. (False!) A $\Delta$-complex $X$ is a **refinement** of another $\Delta$- complex $Y$ if there is a homeomorphism from $Y$ to $X$ mapping any simplex of $Y$ to a subcomplex of $X$. If $X$ is a refinement of $Y$, one can easily prove that $H_*(X) = H_*(Y)$. Thus, if we can prove that any two homeomorphic $\Delta$-complexes have a common refinement, this will prove the theorem. This is the (in)famous **Hauptvermutung** which we alluded to in the lecture notes for surfaces and 3-manifolds, which turns out to be *false* for dimensions four of higher. □

In order to circumvent the difficulty in this false proof, a more general formulation of homology, called **singular homology** has been introduced, which gives identical results for $\Delta$-complexes, but can also be applied to spaces that have no triangulations. We will not delve into these technicalities and refer to Hatcher [Hat02] for the appropriate background on singular homology and the equivalence with the simplicial homology under study here.

## 2.3 Examples and the question of the coefficient ring

Morally, the homology groups count the number of holes in each dimension. But the situation is more subtle due to the torsion that may appear in the homology groups. We illustrate this on a few examples.

In order to compute the homology of a surface, the first step is to describe this surface as a $\Delta$-complex. While, this can be definitely be done, this easily gets a bit unwieldy so we will cheat a bit, and work instead directly with polygonal schemes (i.e., with a polyhedral complex). The reader can verify that it gives the same result.

Let us start with the orientable surface $S$ of genus $g$ over the coefficient ring $\mathbb{Z}$, we take a system of loops made of $2g$ loops so that the resulting complex has 1 face, 1 vertex and $2g$ edges. Then the boundary of the vertex is trivial (as is always the case), but the boundary of the edges as well, since every edge is a loop. And in the boundary of the face, every edge appears once with each orientation, so they cancel out and the boundary is trivial as well. Thus the computation ends up being trivial and we have

$H_0(S)=\mathbb{Z}$, $H_1(S)=\mathbb{Z}^{2g}$ and $H_2(S)=\mathbb{Z}$. We observe that the 1-dimensional homology is the *abelianization* of the fundamental group, which we already saw in the lecture notes on minimum weight bases (Proposition 4.2). It is a general fact, true in any dimension, known as the Hurewicz Theorem (see Hatcher [Hat02, Section 4.2]).

Now, for the non-orientable surface $S$ of genus $g$, let us pick a system of loops corresponding to the polygonal scheme $a_1 \dots a_g \bar{a}_1 \dots a_{g-1}^{-} a_g$, which (exercise) is the non-orientable surface of genus $g$. The boundary of the vertex and the edges is still trivial, but now the boundary of the face $f$ is not, since $a_g$ appears twice with the same orientation. Thus we have $\partial f = 2a_g$. Thus there are no non-trivial 2-cycles. The space of 1-dimensional cycles is generated by $a_1, \dots a_g$ and the space of 1-dimensional boundaries is generated by $2a_g$, thus $H_1(S)$ is isomorphic to $\mathbb{Z}^{g-1} \oplus \mathbb{Z}_2$. Thus there is a 1-dimensional "hole" that disappears when taken twice! Once again, one can verify that $H_1(S)$ is the abelianization of $\pi_1(S)$.

*Exercise* 2.3. Verify that the homology group is the same when computed with a canonical polygonal scheme $a_1 a_1 a_2 a_2 \dots a_g a_g$.

This is also a good illustration of the role of the coefficient ring. The reader can verify that for the orientable surface of genus $g$, taking the ring $\mathbb{Z}_2$ instead of $\mathbb{Z}$ yields $H_0(S) = \mathbb{Z}_2$, $H_1(S) = \mathbb{Z}_2^{2g}$ and $H_2(S) = \mathbb{Z}_2$, so it makes virtually no difference. But for the non-orientable surface, we obtain $\partial f = 2a_g = 0$ since $2 = 0$ in $\mathbb{Z}_2$. Thus there is a 2-dimensional cycle, and the homology groups are now $H_0(S)=\mathbb{Z}_2$, $H_1(S)=\mathbb{Z}_2^g$ and $H_2(S)=\mathbb{Z}_2$. In some sense, the coefficient ring $\mathbb{Z}_2$ prevents us from seeing the torsion that was detected with $\mathbb{Z}$.

Finally, let us try with the ring $\mathbb{Q}$. Now, $\partial f = 2a_g$ is non-trivial, but $\mathbb{Q}^g/2\mathbb{Q}$ is isomorphic to $\mathbb{Q}^{g-1}$, so we have yet another result: similarly the ring $\mathbb{Q}$ does not see the torsion. In some sense, the homology over $\mathbb{Z}$ is the one that contains the most information, this can be formalized in the *universal coefficient theorem* (see Hatcher [Hat02, Section 3.A]), which provides somewhat intricate algebraic constructions to deduce the homology groups over any ring from those over $\mathbb{Z}$.

## 2.4 Betti numbers and Euler-Poincaré formula

When the homology is taken with the coefficient ring $R=\mathbb{Z}$, the homology groups are finitely generated abelian groups, which can be decomposed as a product of cyclic groups:

$$H_k(X) = \mathbb{Z}^{\beta_k(X)} \times \prod_i (\mathbb{Z}/d_i\mathbb{Z})$$

for some integers $\beta_k$ and $1 \le d_1 \le d_2 \le \dots \le d_m$ where each integer $d_i$ is a divisor of its successor $d_{i+1}$. Furthermore this decomposition is unique. The **rank** $\beta_k(X)$ of the free component of $H_k(X)$ is called the $k$**th Betti number** of $X$. The following formula is a wide-reaching generalization of the Euler formula we saw for planar and surface-embedded graphs.

**Theorem 2.4** (Euler-Poincaré formula)**.** *Let $X$ be a finite $\Delta$-complex of dimension $n$, and let $n_i$ be the number of simplices in dimension $i$, then*

$$\sum_{i=0}^{n}(-1)^i\beta_i(K)=\sum_{i=0}^{k}(-1)^i n_i.$$

PROOF. We have $\beta_i(X)=\text{rank}(H_i(X))=\text{rank}(Z_i(X))-\text{rank}(B_i(X))$, and by the rank formula, $\text{rank}(Z_i(X))=n_i-\text{rank}(\text{Im}(\partial_i))$, thus

$$\begin{aligned}\sum_{i=0}^{k}(-1)^i\beta_i(K) &= \sum_{i=0}^{k}(-1)^i(n_i-\text{rank}\,\text{Im}(\partial_i)-\text{rank}\,\text{Im}(\partial_{i+1}))\\ &= \sum_{i=0}^{k}(-1)^i n_i-(-1)^k\text{rank}\,\text{Im}(\partial_{n+1})-\text{rank}\,\text{Im}(\partial_0)\end{aligned}$$

which concludes the proof since the boundaries $\partial_0$ and $\partial_{n+1}$ are empty. □

The quantity $\chi(X)=\sum_{i=0}^{n}(-1)^i\beta_i(K)=\sum_{i=0}^{k}(-1)^i n_i$ is called the **Euler characteristic** of $X$. Since the homology groups are a topological invariant, so is the Euler characteristic. The reader can cross-check with the examples in the previous subsection that for the case of graphs cellularly embedded on surfaces, we recover the Euler formula.

**Remark:** One could also define the Betti numbers as the alternate sum of the ranks of the homology groups obtained when taking a field of characteristic zero for the coefficient ring. By the aforementioned universal coefficient theorem, the two definitions coincide, but note that the hypothesis of characteristic zero is necessary here, as the computation of the $\mathbb{Z}_2$ homology groups of non-orientable surfaces illustrate.

## 2.5 Homology as a functor

We saw that the fundamental group of a surface not only associates a group to the surface, or more generally a topological space, but also associates a group morphism to every continuous maps: it is a *functor* from the category of topological spaces to the category of groups. The same can be said for the homology, which maps the category of topological spaces to the one of modules, or vector spaces if the coefficient ring is a field. We will see how it works for the restricted case of $\Delta$-complexes. A **simplicial map** $f:K\to L$ between two complexes $K$ and $L$ is a map that sends the vertices of $K$ to the vertices of $L$ and the simplex on the vertices $s_0,\dots,s_k$ to the simplex on the vertices $f(s_0,\dots,s_k)$. An example of a simplicial map is the inclusion map for a $\Delta$-complex $K$ included in another $\Delta$-complex $L$.

One would like to extend by linearity a simplicial map into a map on the chains of the complex, but the orientations get in the way. Thus we define, for a simplicial map $f$, another map $f_\#$ that maps $[s_0,\dots,s_k]$ to $[f(s_0),\dots,f(s_k)]$ if the restriction of $f$ to $\{s_0,\dots,s_k\}$ is injective, and 0 otherwise. Now this map can be extended by linearity to the chains of the complexes $K$ and $L$, and it verifies $\partial_k\circ f_\#=f_\#\circ\partial_k$, thus it is a **morphism of chain complexes**. This property ensures that the maps $f_\#$ can be quotiented by the boundary groups: indeed, $f_\#(a+\partial b)=f_\#(a)+\partial(f_\#(b))$, so that the image of $a$ and $a+\partial b$ is the same when quotiented by the boundary space. This allows to define a map $f_\#:H_*(K)\to H_*(L)$, which can also be denoted by $H_*(f)$.

*Exercise* 2.5. Check that $H_*(Id_K) = Id_{H_*(K)}$ and that if $f : K \to L$ and $g : L \to M$ are two simplicial maps, then $H_*(g) \circ H_*(f) = H_*(g \circ f)$. This property is called the **covariance** of the homology functor.

# 3 Homology computations

## 3.1 Over a field

One of the perks of homology is that it is a topological invariant living in the realm of modules, vector spaces or finitely generated abelian groups (depending on the choice of the coefficient rings). Unlike general groups where most problems are undecidable, these algebraic structures are very convenient to exploit in terms of computation. For instance, when the coefficient ring is a field like $\mathbb{Z}_2$, the spaces of chains, boundaries, cycles, and thus the homology groups are all vector spaces. In particular, computing the homology groups in this case amounts to computing kernels and images of explicit operators (the boundaries), and this boils down to linear algebra. For example, using Gaussian elimination, one can easily compute the homology groups in polynomial time in this case, and faster techniques for matrix multiplication allow us to do this even faster.

## 3.2 Computation of the Betti numbers: the Delfinado-Edelsbrunner algorithm

In order to compute the Betti numbers, there is a conceptually simple algorithm due to Delfinado and Edelsbrunner [DE95] which allows to bypass the use of linear algebra in low dimensions. The idea is to add the simplices of a $\Delta$-complex one by one, i.e., to consider a $\Delta$-complex $K$ as a sequence of inclusions (technically, a **filtration**) $K_1 \subseteq K_2 \ldots \subseteq K_m = K$, and to compute the Betti numbers incrementally.

**Proposition 3.1.** *Let $K$ and $K'$ be two $\Delta$-complexes such that $K' = K \cup \sigma$ where $\sigma$ is a $k$-simplex. If the boundary of $\sigma$ in $K'$ is a boundary in $K$, we have*

$$\beta_i(K') = \begin{cases} \beta_i(K)+1 \text{ if } i = k \\ \beta_i(K) \text{ otherwise.} \end{cases}$$

*Otherwise,*

$$\beta_i(K') = \begin{cases} \beta_i(K)-1 \text{ if } i = k-1 \\ \beta_i(K) \text{ otherwise.} \end{cases}$$

PROOF. Let us denote with a prime the objects related to $K'$. The chain complexes of $K$ and $K'$ are identical except for the part $C_k \to^{\partial_k} C_{k-1}$, thus $\beta'_i = \beta_i$ for $i \neq k, k-1$.

If $\partial'_k \sigma$ is a boundary in $K$, i.e., $\partial'_k \sigma \in \operatorname{Im} \partial_k$, then $\operatorname{Im} \partial'_k = \operatorname{Im} \partial_k$. Thus $\beta'_{k-1} = \beta_{k-1}$, and

$$\operatorname{rank} \ker \partial'_k = \operatorname{rank} C'_k - \operatorname{rank} \operatorname{Im} \partial'_k = \operatorname{rank} C_k + 1 - \operatorname{rank} \operatorname{Im} \partial_k = \operatorname{rank} \ker \partial_k + 1.$$

Thus, $\beta'_k = \beta_k + 1$.

In the other case, we have $\operatorname{rank}\operatorname{Im}\partial'_k = \operatorname{rank}\operatorname{Im}\partial_k + 1$ and $\ker\partial'_k = \ker\partial_k$, which gives similarly $\beta'_{k-1} = \beta_{k-1} - 1$ and $\beta'_k = \beta_k$. □

This proposition allows to compute the Betti numbers of a $\Delta$-complex inductively, provided one can test whether the boundary of each newly added simplex was already a boundary. In low dimensions, this is easy: a 0-dimensional complex, i.e., a vertex has no boundary, and a 1-dimensional complex, i.e., an edge, has two vertices as its boundary. These were already a boundary if and only if they are in the same connected component of $K$, which can be tested easily (or not so easily but very efficiently using a Union-Find data structure). Some extensions to 2 and 3-dimensional complexes embedded in $\mathbb{S}^3$ are discussed in the article of Delfinado and Edelsbrunner [DE95], and for the general case this test can be done using the linear algebraic machinery alluded to above.

## 3.3 Over the integers: the Smith-Poincaré reduction algorithm

When the coefficient ring is not a field, one can still compute the homology groups, but this requires slightly more advanced techniques, which we now introduce in the paradigmatic case of the integers $\mathbb{Z}$. The **Smith-Poincaré reduction algorithm** is a variant of Gaussian elimination, tailored to deal with the integers instead of the reals.

The **Smith normal form** of an $r \times c$ integer matrix $M$ is the description of $M$ as a product $M = S\tilde{M}T$, where $S$ is an invertible integral $r \times r$ matrix, $T$ is an invertible integral $c \times c$ matrix and $\tilde{M}$ is an integral $r \times c$ matrix with only diagonal coefficients and each coefficient is a multiple of the previous one (some diagonal coefficients can be zero, but by this condition they have to be the last one). Throughout this paragraph, we will use without mention the well known connections between the multiplication by invertible matrices and the elementary operations on rows and columns. Note the similarity with Gaussian elimination, which consists of a similar factorization but without the constraint that the matrices are integral.

Let us assume that we put all the boundary operators $\partial_k$ in Smith normal form, with diagonal elements $d_{k_1}, \ldots, d_{k_{m_k}}$, then the boundary, cycle, and homology groups are as follows: $Z_k = \mathbb{Z}^{n_k - m_k}$, $B_k = \mathbb{Z}^{m_{k+1}}$ and

$$H_k = \mathbb{Z}^{n_k - m_k - m_{k+1}} \oplus \bigoplus_{i=1}^{m_{k+1}} \mathbb{Z}_{d_{k_i}},$$

where it is understood that $\mathbb{Z}_1$ is trivial and is to omitted from this decomposition.

To prove the existence of a Smith normal form, we start with the following preparatory lemma:

**Lemma 3.2.** *There exist integral matrices $S'$ and $T'$ such that $M = S'\tilde{M}T'$ and in $\tilde{M} = (m'_{ij})$, all the $m'_{ij}$ are multiple of $m'_{11}$.*

Proof. We can assume that $M$ is non-empty, otherwise the lemma is trivial. Let $m_{ij}$ be the coefficient with the smallest absolute value. The proof is an induction on this absolute value.

If all the other coefficients are a multiple of $m_{ij}$ (which includes the base case of the induction $m_{ij} = 1$), we can put it in the top-left corner using permutations of the rows and the columns, which translate into permutation matrices for $S$ and $T$.

Otherwise, there is some $m_{kl}$ that is not a multiple of $m_{ij}$. If $k = i$ or $j = l$, doing the Euclidean division of $m_{kl} = \lambda m_{ij} + \alpha$, and subtracting $\lambda$ times the ith/jth row/column to the kth/lth row/column, we have $\alpha < m_{ij}$ and we proceed by induction.

Finally, if all the coefficients on the ith row and the kth column are multiples of $m_{ij}$, but some $m_{kl}$ persists in not being one, then we have $m_{il} = \lambda m_{ij}$ for some $\lambda$. Subtracting $\lambda - 1$ times the jth column to the lth column, $m_{il}$ gets transformed into $\alpha$, and either $|m_{kl}| < \alpha$ and we proceed by induction, or we are in the previous case. □

We can now prove the theorem

**Theorem 3.3.** *Every integral matrix can be decomposed in a Smith normal form.*

PROOF. By the previous lemma, we can transform the top left element into one that divides all the other ones. Then, this element can be used to kill all the non-diagonal elements in the first row and column, and to obtain the matrix

$$\begin{pmatrix} m_{11} & 0 & \dots & 0 \\ 0 & & & \\ \dots & & M' & \\ 0 & & & \end{pmatrix}$$

where all the elements of $M'$ are multiple of $m_{11}$. Then one can further reduce $M'$ by induction. □

*Exercise* 3.4. Show that the Smith normal form of a matrix is unique.

The proof above yields an easily implementable algorithm to compute the Smith normal form, and thus the integral homology groups of a $\Delta$-complex. However, the complexity of this algorithm is not so good: the size of the integers involved in the computations may easily explode – it is often hidden under the rug, but the same problem arises with the usual Gaussian elimination, which can be circumvented using the Bareiss algorithm (see for example the book of von zur Gathen and Gerhard [VZGG13]). More careful algorithms can be used to control the size of the integers and compute a Smith normal form in polynomial time, we refer to the survey of Dumas et al. [DHSW03].

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