

# Topology of Combinatorial Graphs

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## 1 What Is a Graph?

Graphs are among the most ubiquitous objects in Computer Science. Still, there might be as many formal definitions of a graph as there are books on the subject. This is even the case in the more formalized subfield of algebraic graph theory. For instance, Biggs starts his book on algebraic graph theory [Big94] with

### Basic definitions and notations

Formally, a *general graph*  $\Gamma$  consists of three things: a set  $V\Gamma$ , a set  $E\Gamma$  and an incidence relation, that is, a subset of  $V\Gamma \times E\Gamma$ . An element of  $V\Gamma$  is called a *vertex*, an element of  $E\Gamma$  is called an *edge*, and the incidence relation is required to be such that an edge is incident with either one vertex (in which case it is a *loop*) or two vertices.

While Godsil and Royle [GR01] begin with

## 1.1 Graphs

A *graph*  $X$  consists of a *vertex* set  $V(X)$  and an edge set  $E(X)$ , where an edge is an unordered pair of distinct vertices of  $X$ .

We advocate the following universal definition (see e.g. [Ser77, Sec. 2.1])

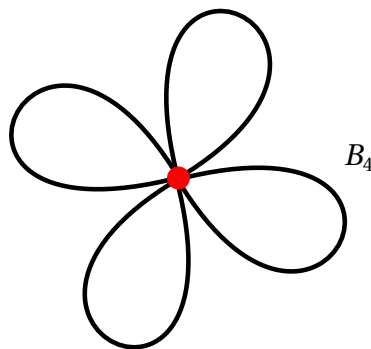
**Definition 1.1.** A **graph** is a quadruple  $G = (V, A, o, \iota)$  where

- $V$  is a set whose elements are called **vertices**,
- $A$  is a set whose elements are called (oriented) **arcs**,
- $o : A \rightarrow V$  is a map that sends each arc  $a$  to its **origin** vertex  $o(a)$ .
- $\iota : A \rightarrow A$  is a fixed point free involution that sends each arc to its **inverse** arc. We usually write  $a^{-1}$  for  $\iota(a)$ .

The origin is also called the **tail** of an arc and the inverse is called the **opposite**.

A (non-oriented) **edge** is a pair  $\{a, a^{-1}\}$ . The origin of  $a^{-1}$  is the **destination**, or **head**, of  $a$ . The tail and head of an edge are its two **endpoints** to which the edge is **incident**. Because  $\iota$  has no fixed point the set of arcs is the disjoint union  $A = A_+ \cup \iota(A_+)$  for some  $A_+ \subset A$ . The set of edges is thus in bijection with  $A_+$ . Fixing  $A_+$  defines a **default orientation** of the edges. *We will assume this default orientation given once for all for each graph in this document.*

*Example 1.2.* A graph with a single vertex is called a **bouquet of circles**. The bouquet of circles with  $n$  edges is denoted by  $B_n$ .



Following Serre [Ser77] “there is an evident notion of morphisms for graphs”. Serre defines a morphism as two maps, one between the vertex sets and one between the arc sets, that “commute” with the origin and inverse maps. For this definition, a non-loop edge contraction would not be a morphism. We thus find more convenient the following slightly modified definition.

**Definition 1.3.** A **morphism** from a graph  $(V, A, o, \iota)$  to a graph  $(V', A', o', \iota')$  is a map  $f: V \cup A \rightarrow V' \cup A'$  such that  $f(V) \subset V'$  and  $f$  commutes with the origin and inverse maps, i.e.,  $f \circ o = o' \circ f$  and  $f \circ \iota = \iota' \circ f$ , where by convention the origin and inverse maps are the identity on the vertex sets.

We will often denote by  $V(G)$  and  $E(G)$  the respective sets of vertices and edges of a graph  $G$ . Note that the number  $|E(G)|$  of edges is half the number of arcs of  $G$  when those numbers are finite. **Subgraphs** are defined in the obvious way by inclusion of the sets of vertices and edges and by restrictions of the origin and inverse maps. Note that the being a subgraph induces an (inclusion) morphism.

## 1.1 Basic operations on graphs

Let  $G = (V, A, o, \iota)$  be a graph, and let  $e = \{a, a^{-1}\}$  be an edge of  $G$ .

**Definition 1.4.** The **contraction** of  $e$  in  $G$  transforms  $G$  to the graph  $G/e = (V', A', o', \iota')$  where  $V' = V/\{o(a) = o(a^{-1})\}$ ,  $A' = A \setminus e$  and  $o'$  and  $\iota'$  are the obvious restrictions of  $o$  and  $\iota$  with the identification of  $o(a)$  and  $o(a^{-1})$ . If  $e$  has a degree one endpoint, the contraction is called an **elementary retraction**.

It is an easy exercise to check that the edge contraction is a graph morphism and that the contractions of edges commute. See Example 1.12 below. More generally,

**Definition 1.5.** If  $E' = B \cup \iota(B)$  is a subset of edges of  $G$ , the contraction of  $E'$  in  $G$  is the graph  $G/E' = (V', A', o', \iota')$  where  $A' = A \setminus (B \cup \iota(B))$  and  $V' = V/\approx$ , where  $\approx$  is the transitive and reflexive closure of the relation  $\bigcup_{b \in B} (o(b), o \circ \iota(b))$ , and  $o'$  and  $\iota'$  are defined in the obvious way.

When  $H$  is a subgraph of  $G$  we also write  $G/H$  for the contractions in  $G$  of all the edges of  $H$ .

**Definition 1.6.** The **deletion of an edge**  $e$  of  $G$  transforms  $G$  to the graph  $G - e = (V, A', o', \iota')$  where  $A' = A \setminus e$  and  $o'$  and  $\iota'$  are the obvious restrictions of  $o$  and  $\iota$ . Similarly, we define the **deletion of a vertex**  $v$  as the graph  $G - v$  with  $v$  and all the incident edges removed.

**Definition 1.7.** The **elementary subdivision** of  $e$  in  $G$  transforms  $G$  to the graph  $S_e G = (V', A', o', \iota')$  where  $V' = V \cup \{x\}$ ,  $A' = A \cup \{a', a'^{-1}\}$  for some new elements  $x, a', a'^{-1}$  not in  $V \cup A$ . The maps  $o'$  and  $\iota'$  are defined in the obvious way. In particular,  $S_e G / \{a', a'^{-1}\} = G$ . A **subdivision** of  $G$  is the result of a finite sequence of elementary subdivisions.

Two graphs are **combinatorially equivalent** if they have isomorphic subdivisions. Intuitively, two graphs are combinatorially equivalent if they have homeomorphic realizations (to be defined). An **invariant** for graphs is a property, usually a functor, that is invariant under combinatorial equivalence. This will be the case for the fundamental group, the homology or cohomology groups as defined in the next sections.

## 1.2 Paths and trees

**Definition 1.8.** A **path** in  $G$  is a finite alternating sequence of vertices and arcs of  $G$  of the form  $(v_0, a_1, v_1, a_2, \dots, a_k, v_k)$  such that the tail and head of  $a_i$  are respectively  $v_{i-1}$  and  $v_i$ . The vertices  $v_0$  and  $v_k$  are the **endpoints** of the path. The integer  $k$  is the **length** (or size) of the path. When  $k = 0$ , the path is said to be **constant**. It is denoted by 1 when the vertex  $v_0$  is implicit. The **inverse path** of  $\gamma = (v_0, a_1, v_1, a_2, \dots, a_k, v_k)$  is the path  $\gamma^{-1} = (v_k, a_k^{-1}, v_{k-1}, a_{k-1}^{-1}, \dots, a_1^{-1}, v_0)$ . A path is **simple** if it has no repeated vertices. It is **closed** if its first and last vertex coincide. For non-constant paths, the vertices are redundant and we usually write  $(a_1, a_2, \dots, a_k)$  or  $a_1 \cdot a_2 \cdot \dots \cdot a_k$  for  $(v_0, a_1, v_1, a_2, \dots, a_k, v_k)$ . A path is also called a **walk** and a closed path is also called a **loop**. The first vertex of a loop is its **basepoint**. We reserve the term **circuit** to a closed path without fixing its basepoint. Formally, a circuit is a class of closed paths related by circular permutations of their arcs. A circuit is **simple** if all its paths are simple.

**Definition 1.9.** A graph is **connected** if every pair of its vertices can be joined by a path in the graph. Note that we may require that the path is simple in this definition without changing the property of being connected. A graph without simple closed path is a **forest**. A connected forest is a **tree**.

It is easily seen from the definition that every two vertices in a tree are connected by a unique simple path. As far as topological properties are concerned the most fundamental properties of a graph are recorded in the next two lemmas.

**Lemma 1.10.** *Any subtree of a graph  $G$  can be extended to a maximal subtree under inclusion.*

**PROOF.** This is clear if  $G$  is finite as it contains a finite number of subtrees. Otherwise, consider the set of subtrees of  $G$  containing the given subtree, say  $T$ , ordered by the subgraph relation (*i.e.*, by inclusion). It is easily checked that the union of the trees in any ascending chain of subtrees is again a subtree containing  $T$ . It thus provides an upper bound to the ascending chain. It follows from Zorn's lemma that the set of subtrees of  $G$  containing  $T$  has a maximal element. This concludes the proof. A more "constructive" proof makes use of the axiom of choice to build a shortest path tree inductively.  $\square$

A subgraph of  $G$  is **spanning** if it contains all the vertices in  $G$ .

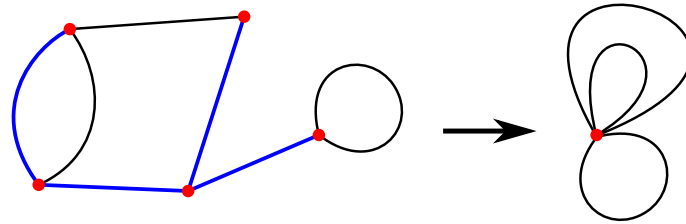
**Corollary 1.11.** *Every connected graph has a spanning tree.*

**PROOF.** Let  $G$  be a connected graph. By Lemma 1.10,  $G$  has a maximal subtree  $T$  extending a single vertex of  $G$ , say  $v$ . We show that  $T$  is spanning. Suppose by way of contradiction that some vertex  $w$  of  $G$  is not a vertex of  $T$ . By connectedness of  $G$  there is a simple path from  $v$  to  $w$  in  $G$ . Let  $x$  be the last vertex in  $T$  along this path and let  $a$  be the arc with origin  $x$  along the path. Then, adding the edge  $\{a, a^{-1}\}$  to  $T$  we obtain a subtree of  $G$  larger than  $T$ , in contradiction with the maximality of  $T$ .  $\square$

If  $T$  is a spanning tree of the connected graph  $G$ , any edge of  $G$  that is not in  $T$  is a **chord**. Note that a chord may or may not be a loop edge.

*Example 1.12.* If  $T$  is a subtree of a  $G$ , then there is a morphism  $c_T : G \rightarrow G/T$  contracting the edges of  $T$ . If  $T$  is finite,  $c_T$  is a composition of edge contractions. Otherwise,  $G/T$  is a direct limit of such contractions.

*Example 1.13.* When  $T$  is a spanning tree,  $G/T$  is a bouquet of circles whose edges correspond to the chords of  $T$  in  $G$ .

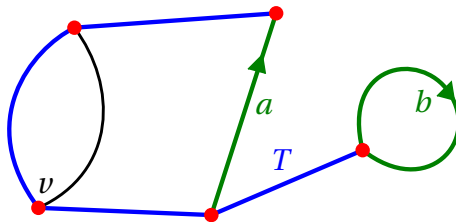


*Example 1.14.* If  $T$  is a spanning tree of the connected graph  $G$ , we denote by  $T[v, w]$  the unique shortest path in  $T$  from  $v$  to  $w$ . Any arc  $a$  of  $G$  determines a loop with basepoint  $v$ :

$$T[v, a] := T[v, o(a)] \cdot a \cdot T[o(a^{-1}), v]$$

and a circuit

$$T[a] := a \cdot T[o(a^{-1}), o(a)]$$



Here,  $T[a] = T[v, a]$  but  $T[b] = (b) \neq T[v, b]$ .

## 2 Path homotopy in graphs

**Definition 2.1.** A **spur** is a subsequence of the form  $(a, a^{-1})$  in a path. Adding or removing a spur in a path is called an **elementary homotopy**. A **free elementary homotopy** is an elementary homotopy applied to any of the path representatives of a circuit. **Homotopy** is the equivalence relation generated by elementary homotopies. Likewise, **free homotopy** is the reflexive and transitive closure of free elementary homotopies. We write  $\gamma \sim \lambda$  if  $\gamma$  and  $\lambda$  are homotopic paths and  $\gamma \overset{free}{\sim} \lambda$  when they are freely homotopic circuits. A path is **reduced** if it does not contain any spur. Similarly, a circuit without spur is said **cyclically reduced**. A path or circuit (freely) homotopic to a constant path is said to be **contractible**. If the last vertex of a path  $\gamma$  coincides with the first vertex of a path  $\lambda$ , their **concatenation** is the path  $\gamma \cdot \lambda$  whose arc sequence is the arc sequence of  $\gamma$  followed by the arc sequence of  $\lambda$ .

Note that concatenation is compatible with the homotopy relation:  $\gamma \sim \gamma'$  and  $\lambda \sim \lambda' \implies \gamma \cdot \lambda \sim \gamma' \cdot \lambda'$ . Adding that  $\gamma \cdot \gamma^{-1} \sim 1$  for all paths  $\gamma$  implies the next proposition.

**Proposition 2.2.** *Let  $v$  be a vertex of  $G$ . The set of homotopy classes of loops with basepoint  $v$  is a group for the law of path concatenation, with the constant path for the unity. It is called the **fundamental group** of  $G$  based at  $v$  and denoted by  $\pi_1(G, v)$ . The free homotopy classes are the conjugacy classes in this group.*

**Lemma 2.3.** *Every homotopy class has a unique reduced path. Similarly, every free homotopy class has a unique cyclically reduced circuit.*

PROOF. Let  $\gamma \sim \lambda$  with  $\gamma$  and  $\lambda$  reduced. Choose a sequence  $\gamma = \mu_0 \sim \dots \sim \mu_k = \lambda$  of elementary homotopies such that the total length  $\sum_i |\mu_i|$  is minimal. Note that  $k \neq 1$  since  $\gamma$  and  $\lambda$  are both reduced. If  $k > 1$ ,  $\mu_1$  and  $\mu_{k-1}$  are longer than  $\gamma$  and  $\lambda$  respectively. Let  $i \in \{1 \dots k-1\}$  be such that  $|\mu_i|$  is maximal. Then,  $\mu_{i-1}$  and  $\mu_{i+1}$  are each obtained from  $\mu_i$  by removing a spur. The two spurs may either be disjoint or equal, or may overlap in  $\mu_i$ . In each case, it is a simple exercise to check that the total length of the sequence of elementary homotopies can be reduced. This contradicts the minimality of the chosen sequence. It follows that  $k = 0$ , so that  $\gamma = \lambda$  as desired. The proof of uniqueness for the case of free homotopy is similar.  $\square$

*Exercise 2.4.* Let  $\gamma$  be a path from a vertex  $v$  to a vertex  $w$  in  $G$ . Prove that the map  $\lambda \mapsto \gamma \cdot \lambda \cdot \gamma^{-1}$  taking a loop with basepoint  $w$  to a loop with basepoint  $v$  induces an isomorphism from  $\pi_1(G, w)$  to  $\pi_1(G, v)$ . What is this isomorphism when  $\gamma$  is a loop?

When  $G$  is connected, Exercise 2.4 allows us to speak of the fundamental group of  $G$ , defined up to isomorphism, without referring to its basepoint.

**Proposition 2.5.** *The fundamental group of a bouquet of circles is a free group over its edge set.*

PROOF. Let  $B$  be bouquet of circles with arc set  $A$ . Recall that we can write  $A = A_+ \cup \iota(A_+)$ , so that the set of edges can be identified with  $A_+$ . We denote by  $F(A_+)$  the free group generated by  $A_+$ . Since  $B$  has a single vertex  $\bullet$ , each arc  $a$  is a loop  $(a)$ . Obviously, the set of loops  $\{(a)\}_{a \in A_+}$  generates  $\pi_1(B, \bullet)$ . The map  $a \mapsto (a)$  extends uniquely, by the universal property of free groups, to a group morphism  $F(A_+) \rightarrow \pi_1(B, \bullet)$  that is onto by the preceding obvious remark. Since elementary homotopies correspond to free elementary reductions of words (of the type  $uaa^{-1}v \mapsto uv$ ), the kernel of this morphism is trivial and  $\pi_1(B, \bullet) \simeq F(A_+)$ .  $\square$

*Remark 2.6.* To keep the notations light, we will often identify a loop with its homotopy class.

**The  $\pi_1$  functor** A graph morphism  $f : G \rightarrow G'$  can be extended to send a path, loop or circuit  $(a_1, a_2, \dots, a_k)$  of  $G$  to a path, loop or circuit  $(f(a_1), f(a_2), \dots, f(a_k))$  of  $G'$ , ignoring the  $f(a_i)$  that are vertices. This extension commutes with path concatenation, while homotopic paths and freely homotopic circuits are sent to homotopic paths and freely homotopic circuits respectively. It follows that

**Lemma 2.7.** *A graph morphism  $f : (G, v) \rightarrow (G', f(v))$  induces in a natural way a group morphism  $f_* : \pi_1(G, v) \rightarrow \pi_1(G', f(v))$ , i.e., if  $G \xrightarrow{f} G' \xrightarrow{g} G''$  are two morphisms, then  $(fg)_* = f_*g_*$ .*

The correspondences  $(G, v) \mapsto \pi_1(G, v)$  and  $f \mapsto f_*$  thus define a functor from the category of connected pointed graphs to the category of groups.

*Exercise 2.8.* Prove that an edge contraction of a connected graph induces an isomorphism of fundamental groups if and only if its endpoints are distinct.

**Theorem 2.9.** *Let  $T$  be a spanning tree of a connected graph  $G$ . For any vertex  $v$  of  $G$ ,  $\pi_1(G, v)$  is isomorphic to the free group on the set of chords of  $T$  in  $G$ .*

**PROOF.** We again write  $A = A_+ \cup \iota(A_+)$  for the set of arcs of  $G$  and we will freely identify a subset of edges with a subset of  $A_+$  when convenient. We denote by  $C$  the set of chords of  $T$  in  $G$ . We observe that any loop  $(a_1, a_2, \dots, a_k)$  is homotopic to the concatenation of loops  $T[v, a_1] \cdot T[v, a_2] \cdots T[v, a_k]$  (See Example 1.14). Since  $T[v, a]$  is contractible whenever  $a$  is in  $T$ , we see that the family  $\Gamma = \{T[v, a]\}_{a \in C}$  generates  $\pi_1(G, v)$ . Each arc of  $C$  appears exactly once in one loop of this family. It follows that  $\Gamma$  only satisfies trivial relations (of the type  $T[v, a] \cdot T[v, a]^{-1} = 1$ ) and is thus a free generating set. Said differently, the map  $a \mapsto T[v, a]$  extends uniquely to a morphism  $F(A_+) \rightarrow \pi_1(G, v)$  whose kernel is the subgroup spanned by the edges of  $T$ . We conclude that  $\pi_1(G, v) \simeq F(A_+)/F(T) \simeq F(C)$ .  $\square$

*Remark 2.10.* From the proof, we note that a basis of  $\pi_1(G, v)$  is given by the loops  $T[v, a]$  when  $a$  runs through the chords of  $T$  in  $G$ . The expression in this basis of the homotopy class of a loop  $\ell$  is obtained as follows. We first take the trace of  $\ell$  over  $C$ , i.e., we discard the arcs of  $T$  in  $\ell$ . We then freely reduce the resulting word on  $C \cup \iota(C)$ , and finally replace each occurrence of  $c$  (resp.  $c^{-1}$ ) by  $T[v, c]$  (resp.  $T[v, c]^{-1}$ ).

**Corollary 2.11.** *If  $G$  is a finite connected graph, its fundamental group is a free group of rank*

$$1 - \chi(G) = 1 - |V(G)| + |E(G)|$$

**PROOF.** From the preceding theorem  $r := \text{rank} \pi_1(G, v)$  is the number of chords of a spanning tree  $T$ , so that  $r = |E(G)| - |E(T)|$ . But  $T$  being a tree we have  $|E(T)| = |V(T)| - 1$  and  $T$  being spanning we have  $|V(T)| = |V(G)|$ . Whence  $r = |E(G)| - (|V(G)| - 1)$ .  $\square$

*Exercise 2.12.* Let  $H$  a connected subgraph of a connected graph  $G$  and let  $v$  be vertex of  $H$ . Prove that the inclusion  $H \hookrightarrow G$  induces a monomorphism  $\pi_1(H, v) \hookrightarrow \pi_1(G, v)$ . (Hint. You may use a direct proof or use Lemma 1.10.)



### 3 Some Elementary Algorithms Related to Homotopy

Here, we examine how to compute a basis of the fundamental group of a finite graph in practice, and how to decide whether a loop is contractible or whether two loops are homotopic. We assume given a finite connected graph  $G = (V, A, o, \iota)$  with a default orientation  $A_+$  and a vertex  $v \in V$ .

#### 3.1 Computing a basis of $\pi_1(G, v)$

By a **basis** we mean a minimal size set of loops whose homotopy classes generate  $\pi_1(G, v)$ . Note that a reduced loop (without spur) has minimal length among all its homotopic loops.

**Lemma 3.1.** *We can compute a basis of  $\pi_1(G, v)$  in time  $O(|A_+| + r|V|)$  where  $r = 1 - |V| + |A_+|$ .*

**PROOF.** We already know from Corollary 2.11 that any basis has  $r$  elements. By the remark following Theorem 2.9, such a basis is provided by the loops  $\{T[v, a]\}$ , for  $a$  a chord of a spanning tree  $T$ . The spanning tree can be computed using a graph traversal such as depth-first search or breadth-first search in  $O(|A_+|)$  time. Each of the  $r$  loops  $\{T[v, a]\}$  can be written down in time proportional to its length  $O(|V|)$ .  $\square$

It should be noted that a basis of  $\pi_1(G, v)$  does not necessarily arise from the chords of a spanning tree. However, a shortest basis – that is a basis minimizing the total length of its loops – indeed arises this way. To see this we state a preliminary lemma.

**Lemma 3.2.** *Let  $F$  be a free group over  $(x_1, x_2, \dots, x_n)$ . For any base  $(u_1, u_2, \dots, u_n)$  of  $F$  there exists a permutation  $\sigma$  of  $[1, n]$  such that each  $x_i$  appears in the reduced expression of  $u_{\sigma(i)}$  in terms of the  $x_j$ .*

**PROOF.** The automorphism of  $F$  defined by  $x_i \mapsto u_i$ ,  $i \in [1, n]$ , quotients to an automorphism  $f$  of its abelianized group  $F/[F, F]$  which is a free abelian group of rank  $n$ . The map  $f$  can thus be seen as an automorphism of  $\mathbb{Z}$ -module whose matrix  $(c_{ij})$  with respect to the basis formed by the cosets of the  $x_i$  – so that  $c_{ij}$  is the cumulative exponents of  $x_i$  in  $u_j$  – has a non-zero determinant. It follows that at least one term  $\prod_{i \in [1, n]} c_{i\sigma(i)}$  of the usual Leibnitz expansion of the determinant must be non-zero. This implies the lemma.  $\square$

**Proposition 3.3.** *The basis of  $\pi_1(G, v)$  associated to a breadth-first-search tree from  $v$  is a shortest basis.*

**PROOF.** Let  $T$  be a breadth-first-search tree from  $v$ . In particular, for any arc  $a$ ,  $T[v, a]$  is a shortest loop with base  $v$  through  $a$  in  $G$ . We denote by  $c_1, c_2, \dots, c_r$  the chords of  $T$  in  $G$ . Let  $B$  be another basis. According to the previous lemma, the elements of  $B$  can be ordered in a such a way that its  $i$ th element  $b_i$  contains  $T[v, c_i]$  in its reduced expression in terms of the  $T[v, c_j]$ . It follows that  $b_i$  goes through  $c_i$ , hence is longer than  $T[v, c_i]$ .  $\square$



*Remark 3.4.* If the edges of  $G$  are positively weighted, a shortest basis is a basis whose total weight is minimal. Then, Proposition 3.3 still holds if we replace the breadth-first-search tree by a shortest path tree with respect to the distance given by the weights.

**Open question:** Can we characterize which source vertices  $v$  in  $G$  lead to the shortest of the shortest bases?

### 3.2 Homotopy test

**Proposition 3.5.** *We can test whether any two loops  $\ell$  and  $\ell'$  are homotopic in  $O(|\ell|+|\ell'|)$  time.*

PROOF Let  $\lambda$  and  $\lambda'$  be obtained respectively from  $\ell$  and  $\ell'$  by maximally removing spurs in them. By Lemma 2.3,  $\ell \sim \ell'$  if and only if  $\lambda$  and  $\lambda'$  are equal. Note that those reduced forms can be obtained in linear time using a stack to remove spurs in a single scan of  $\ell$  and  $\ell'$ .  $\square$

We have a similar result for free homotopy.

**Proposition 3.6.** *We can test whether any two circuit  $c$  and  $c'$  are freely homotopic in  $O(|c|+|c'|)$  time.*

PROOF Let  $\kappa$  and  $\kappa'$  be obtained respectively from  $c$  and  $c'$  by maximally and cyclically removing spurs in them. By Lemma 2.3,  $c \overset{\text{free}}{\sim} c'$  if and only if  $\kappa$  and  $\kappa'$  are equal up to a cyclic permutation. Those reduced forms can be obtained in linear time from (representatives of)  $c$  and  $c'$  using for each of them a doubly linked list and scanning in both directions to remove spurs from both ends of the list. Remark that  $\kappa$  is a cyclic permutation of  $\kappa'$  if and only if they have the same length and  $\kappa$  is a substring of the square  $\kappa' \cdot \kappa'$ . This last test can be performed in linear time using the Knuth-Morris-Pratt algorithm (see previous lecture).

One may slightly optimize this last step by further reducing the size of  $\kappa$  and  $\kappa'$ . Indeed, as explained in Remark 2.10, we can express them in the basis associated to the chords of a spanning tree of  $G$  by simply discarding the tree edges in  $\kappa$  and  $\kappa'$ . We can then identify the chords with a basis of  $\pi_1(G, v)$  and apply KMP to the reduced expressions of  $\kappa$  and  $\kappa'$  in this basis. This however assumes a precomputation of a spanning tree for  $G$ .  $\square$

## 4 Homology

We now define the cycle group of a graph  $G = (V, A, o, \iota)$  with  $A = A_+ \cup \iota(A_+)$ . The homology of graphs appears in a 1847 paper by Kirchhoff [BLW98, p. 133] concerning electrical networks. Those are modeled as graphs whose edges represent electrical connections each having a resistance  $r_j$  and a voltage source  $E_j$ . Kirchhoff's voltage

law states that the directed sum of the electrical potential differences around a cycle must be zero. Applied to a cycle of a graph, this leads to equations of the form

$$\sum_j r_j I_j = \sum_j E_j$$

where  $j$  runs through the arcs of the cycle, the arc  $j$  being traversed by the current  $I_j$ . If the resistances and sources are known, Kirchhoff explains how to find the minimum number of equations as above, hence the minimum number of cycles, necessary to determine the currents (assuming the Kirchhoff's current law). This minimum is given by the *cyclomatic number* of the graph, which is also the rank of its *cycle space*.

Let  $C_0(G)$  and  $C_1(G)$  be the free abelian groups with basis  $V$  and  $A_+$  respectively. The elements of  $C_i(G)$ ,  $i = 0, 1$  are called ***i-chains***. The **support** of a chain is the set of vertices or arcs with nonzero coefficients; its elements are **contained** in the chain. We also consider the **boundary operator**  $\partial : C_1(G) \rightarrow C_0(G)$  defined by  $\partial a = o(a^{-1}) - o(a)$ . The **homology group of dimension zero** is the quotient

$$H_0(G) = C_0(G) / \text{Im } \partial$$

We simply write  $C_i$  for  $C_i(G)$  when there is no ambiguity on the graph  $G$ .

**Proposition 4.1.**  *$H_0(G)$  is isomorphic to the free abelian group over the set of connected components of  $G$ .*

PROOF Let  $K$  be the set of connected components of  $G$ . Consider the *augmentation map*  $\varepsilon : C_0 \rightarrow \oplus_K \mathbb{Z}$ ,  $c \mapsto \sum_{\kappa \in K} \alpha_\kappa \kappa$  where  $\alpha_\kappa$  is the sum of the coefficients in  $c$  of the vertices belonging to  $\kappa$ . We claim that  $\ker \varepsilon = \text{Im } \partial$ . Indeed, for any arc  $a$  we obviously have  $\varepsilon(\partial a) = 0$ , whence  $\text{Im } \partial \subseteq \ker \varepsilon$ . On the other hand, if  $c = \sum_i \alpha_i v_i \in \ker \varepsilon$  has all its vertices  $v_i$  in a single component, we can join some fixed vertex in this component to each  $v_i$  with a path  $\gamma_i$  and we easily check that  $c = \partial(\sum_i \alpha_i \gamma_i)$ . It follows that  $c \in \text{Im } \partial$  thus proving the claim. We conclude thanks to the surjectivity of the augmentation map that

$$\oplus_K \mathbb{Z} \simeq C_0 / \ker \varepsilon = C_0 / \text{Im } \partial$$

□

We put  $Z_1 := \ker \partial$  and call its elements **cycles**.  $Z_1$  is the **cycle group** of  $G$ . This group is also called the **first homology group** and denoted by  $H_1(G)$ .

Every loop or circuit  $(a_1, a_2, \dots, a_k)$  in  $G$  gives rise to the chain  $\sum_{i=1}^k \varepsilon_i a_i^{\varepsilon_i}$ , where  $\varepsilon_i = 1$  if  $a_i \in A_+$  and  $\varepsilon_i = -1$  if  $a_i \in \iota(A_+)$ . To simplify notations, we use the convention that  $a = -a^{-1}$  whenever  $a \in \iota(A_+)$ . Equivalently, we could define the group of 1-chains as the free group over  $A$  quotiented by the subgroup generated by the "relations"  $a + a^{-1}$ . In both cases, the sum  $\sum_{i=1}^k \varepsilon_i a_i^{\varepsilon_i}$  can be written more simply as  $\sum_{i=1}^k a_i$ . A cycle is said **simple** if it is the sum of the arcs of a simple circuit in  $G$ .

**Lemma 4.2.** *Every cycle is a combination of simple cycles.*

PROOF. Let  $c = \sum_{a \in A_+} \alpha_a a$  be a cycle of  $G$ . The proof is by induction on the size of the support of  $c$ . Let  $H$  be the subgraph induced by the support edges of  $c$ . A vertex of  $H$  cannot have degree one. For otherwise, such a vertex would be contained in the support of  $\partial c$ , contradicting  $\partial c = 0$ . Now,  $H$  being a finite graph, it must contain a simple circuit  $\gamma$ . Let  $\alpha$  be the coefficient in  $c$  of some chosen arc in  $\gamma$ . We conclude by applying the induction hypothesis to the cycle  $c - \alpha\gamma$ .  $\square$

**Corollary 4.3.** *A tree is **acyclic**, i.e., its cycle space is trivial.*

PROOF. From the previous lemma, since a tree has no simple circuit.  $\square$

**Proposition 4.4.** *Suppose  $G$  connected and let  $T$  be a spanning tree of  $G$ . Then,  $H_1(G)$  is isomorphic to the free abelian group generated by the chords of  $T$  in  $G$ .*

PROOF. Let  $C$  be the set of chords of  $T$  in  $G$ . Each arc  $a$  determines a simple cycle  $T[a]$  corresponding to the circuit formed by  $a$  and the unique path in  $T$  from the tail of  $a$  to its origin. Let  $B = \{T[a]\}_{a \in C}$  be the set of simple cycles associated to the chords. The cycles in  $B$  are independent as each chord appears in the support of exactly one of them. It remains to prove that  $B$  is generating. Let  $c = \sum_{a \in T} \alpha_a a + \sum_{e \in C} \beta_e e$  be a cycle in  $G$ . Then,  $c - \sum_{e \in C} \beta_e T[e]$  is a cycle whose support lies in  $T$ . It must be null by the above corollary, i.e.,  $c = \sum_{e \in C} \beta_e T[e]$ .  $\square$

The rank of  $H_1(G)$  is called the **cyclomatic number** or **first Betti number** and denoted by  $\beta_1(G)$ . When  $G$  is connected, remark from Theorem 2.9  $\beta_1(G)$  is also the rank of the fundamental group of  $G$ . In fact

**Theorem 4.5** (Hurewicz).  *$H_1(G)$  is isomorphic to the abelianization of the fundamental group of  $G$ .*

PROOF. Denote by  $\mathcal{L}(G, v)$  the set of loops of  $G$  with basepoint  $v$ . The map  $\mathcal{L}(G, v) \rightarrow H_1(G)$  defined by  $(a_1, a_2, \dots, a_k) \mapsto \sum_{i=1}^k a_i$  is invariant under elementary homotopies and “commutes” with concatenation. It thus defines a morphism  $\varphi : \pi_1(G, v) \rightarrow H_1(G)$ . We just saw that any cycle can be written as a combination of cycles of the form  $T[a]$ , for  $T$  a spanning tree of  $G$ . Noting that  $\varphi(T[v, a]) = T[a]$  in  $H_1(G)$ , it follows that  $\varphi$  is onto. Let  $\gamma = T[v, a_1] \cdot T[v, a_2] \cdots T[v, a_k]$  be any element of  $\pi_1(G, v)$  written over the basis  $\{T[v, a]\}_{a \in C}$ . Then  $\varphi(\gamma) = \sum_{a \in C} n_a T[a]$  where  $n_a$  is the cumulative exponent of  $T[v, a]$  in  $\gamma$ . Hence,  $\gamma \in \ker \varphi$  if and only if all the  $n_a$  cancels. This is exactly saying that  $\gamma$  belongs to the **derived subgroup**  $[\pi_1(G, v), \pi_1(G, v)]$  of  $\pi_1(G, v)$ . We thus have

$$H_1(G) \simeq \pi_1(G, v) / \ker \varphi = \pi_1(G, v) / [\pi_1(G, v), \pi_1(G, v)]$$

$\square$

*Exercise 4.6.* Show that the one dimensional homology of  $G$  is the direct sum of the one dimensional homology of its 2-connected components (the blocks of  $G$ ). (Hint: consider the map sending a cycle to its traces over the 2-connected components of the graph.)

**The homology functor** Let  $f : G \rightarrow G'$  be a graph morphism.  $f$  induces a **chain morphism**  $f_{\#} : C_i(G) \rightarrow C_i(G')$  by setting for  $v \in V(G)$  and  $a \in A(G)$ :

$$f_{\#}(v) = f(v) \quad \text{and} \quad f_{\#}(a) = \begin{cases} 0 & \text{if } f(a) \in V(G') \\ f(a) & \text{otherwise} \end{cases}$$

and by linear extension to chains.

**Proposition 4.7.** *The chain morphism commutes with the boundary operator, i.e.,*

$$f_{\#} \circ \partial = \partial' \circ f_{\#}$$

(We use a prime to denote the boundary operator for  $G'$ .) Hence,  $f$  induces a morphism between homology groups  $f_* : H_i(G) \rightarrow H_i(G')$ ,  $i = 0, 1$ .

PROOF. The commutativity of  $f_{\#}$  with the boundary operator is a direct consequence of the commutativity of morphisms with the origin map. It follows that  $f_{\#}$  sends the kernel and image of  $\partial$  into the kernel and image of  $\partial'$ , respectively. Hence,  $f_{\#}$  descends to a quotient  $f_* : C_0/\text{Im } \partial \rightarrow C'_0/\text{Im } \partial'$  that restricts to a morphism  $f_* : \ker \partial \rightarrow \ker \partial'$ .  $\square$

It is easily checked that the composition of two graph morphisms  $f \circ g$  satisfies  $(f \circ g)_* = f_* \circ g_*$  and that the identity of a graph induces the identity of its homology group. In other words the association of graphs and morphisms to the corresponding homology groups and group morphisms is a functor.

**Homology with other coefficients** We can define homology relatively to any abelian coefficient group  $\Gamma$ . To this end, we define a chain of vertices or arcs as a formal combination with coefficients in  $\Gamma$ . The set of chains is equipped with a group structure induced by the law of  $\Gamma$ . Alternately, these chain groups could be defined by tensoring  $C_0$  and  $C_1$  with  $\Gamma$ . The boundary operator and the homology groups are then defined as for integer coefficients taking into account the new definition of chain groups. The homology with integer coefficients is the most general in the sense that it determines homology over any other group. This is the content of the *universal coefficient theorem* for homology [Hat02, Sec. 3.A]. However, it is often convenient to restrict to other coefficients for computational reasons or to concentrate on specific properties of homology. Common choices for the coefficients include the field of rationals  $\mathbb{Q}$  and the finite cyclic groups  $\mathbb{Z}/p\mathbb{Z}$ . A specific case occurs for  $\Gamma = \mathbb{Z}/2\mathbb{Z} = \{\bar{0}, \bar{1}\}$ . A chain with  $\mathbb{Z}/2\mathbb{Z}$  coefficients can be interpreted as a subset of vertices or edges and the sum of two chains becomes their symmetric difference. A cycle is just a subgraph of  $G$ , each vertex of which has even degree. Such subgraphs are sometimes called *Eulerian*<sup>1</sup>, or *even subgraphs*.

<sup>1</sup>The terminology can be confusing as it is also used to mean a connected Eulerian subgraph

## 5 Cohomology

Cohomology is defined dually to homology. We again consider a graph  $G$  and its chain groups  $C_0(G)$  and  $C_1(G)$ . The **cochain groups** are the dual groups  $C^0(G) = \text{hom}(C_0(G), \mathbb{Z})$  and  $C^1(G) = \text{hom}(C_1(G), \mathbb{Z})$  of linear maps  $C_0(G) \rightarrow \mathbb{Z}$  and  $C_1(G) \rightarrow \mathbb{Z}$ , respectively. We simply write  $C^i$  for  $C^i(G)$  when there is no ambiguity on the underlying graph  $G$ . Viewing the vertices and arcs as elementary chains, we have that  $V(G)$  and  $A_+$  constitute bases of  $C_0$  and  $C_1$ , respectively. It follows that the elements of the cochain groups  $C^0$  and  $C^1$  can be specified by the values they take on the corresponding bases. The elements of  $C^1$  are also called **cocycles**<sup>2</sup>. The dual of the boundary operator is the **coboundary operator**  $\delta : C^0 \rightarrow C^1, f \mapsto f \circ \partial$ . The **coboundary group** is the group  $\text{Im } \delta$ . Its elements are called coboundaries. The **cohomology groups** of  $G$  are

$$H^0(G) = \ker \delta \quad \text{and} \quad H^1(G) = C^1 / \text{Im } \delta$$

When  $G$  is not connected the cohomology groups are (direct) products of the cohomology groups of each component. This follows from the fact that the cochain groups are products of the cochain groups of the components of  $G$  and that the coboundary operator is a product of componentwise coboundaries. We can thus restrict ourselves to connected graphs.

**Lemma 5.1.** *If  $G$  is connected  $H^0(G)$  is infinite cyclic (isomorphic to  $\mathbb{Z}$ ).*

PROOF. An element  $f$  of  $\ker \delta$  is such that  $f(\partial a) = 0$  for any arc  $a$ , i.e.  $f(o(a^{-1})) = f(o(a))$ . By connectivity of  $G$  it follows that  $f$  takes the same value for all the vertices. The kernel of  $\delta$  is thus the set of multiples of the constant map sending each vertex to one.  $\square$

**Lemma 5.2.** *The first cohomology group of a tree is trivial.*

PROOF. Let  $v$  be a vertex of a tree  $T$ . We consider the map

$$\sigma_T : C^1(T) \rightarrow C^0(T), \quad f \mapsto \sigma_T(f) : w \in V(T) \mapsto \sum_{a \in T[v,w]} f(a)$$

We easily check that for any  $f \in C^1(T)$  we have  $\delta \sigma_T(f) = f$ . It follows that  $\text{Im } \delta = C^1(T)$ , i.e.,  $H^1(T)$  is trivial.  $\square$

**Proposition 5.3.** *Let  $T$  be spanning tree of a connected graph  $G$ . Then  $H^1(G)$  is isomorphic to the product of copies of  $\mathbb{Z}$ , with one copy per chord of  $T$  in  $G$ .*

PROOF. Let  $C$  be the set of chords of  $T$  in  $G$ . We view elements of  $\Pi_C \mathbb{Z}$  as functions  $C \rightarrow \mathbb{Z}$ . We consider the group morphism  $\pi : \Pi_C \mathbb{Z} \rightarrow C^1(G) / \text{Im } \delta$  that maps a function  $\phi : C \rightarrow \mathbb{Z}$  to the class of the cocycle  $\pi(\phi) : C_1(G) \rightarrow \mathbb{Z}$  defined for all  $a \in A_+$  by:

$$\pi(\phi)(a) = \begin{cases} \phi(a) & \text{if } a \in C \\ 0 & \text{if } a \in T. \end{cases}$$

<sup>2</sup>For higher dimensional complexes, the cocycle group is the kernel of the coboundary operator  $C^1 \rightarrow C^2$ .

For  $g \in C^1(G)$ , we apply the morphism  $\sigma_T$  of Lemma 5.2 to its restriction  $g|_T$  on  $T$  and view  $\sigma_T(g|_T)$  as a cochain in  $C^0(G)$  since  $V(T) = V(G)$ . Note from Lemma 5.2 that  $\sigma_T(g|_T) = g|_T$ , so that  $g - \delta\sigma_T(g|_T)$  cancels over  $T$ . This last cocycle restricts in turn to a function  $\phi : C \rightarrow \mathbb{Z}$  such that  $\pi(\phi)$  is the cohomology class of  $g - \delta\sigma_T(g|_T)$ , hence of  $g$ . This shows that  $\pi$  is onto. On the other hand,  $\pi(\phi) \in \text{Im } \delta$  implies  $\pi(\phi) = \delta f$  for some  $f \in C^0(G)$ . Because  $T$  is connected and  $\pi(\phi)$  cancels over  $T$ , the cochain  $f$  must be constant on the vertices of  $T$ . Because  $T$  is spanning,  $\delta f$  is also null on  $C$ , whence  $\phi = 0$ . It follows that  $\pi$  is injective, hence an isomorphism.  $\square$

## 6 Some Elementary Algorithms Related to Homology

As for the fundamental group, we examine how to compute a basis of the first homology group of a finite connected graph  $G$ . Following the proof of Proposition 4.4, or by applying Theorem 4.5, the cycles  $T[a]$  when  $a$  runs over the chords of a spanning tree  $T$  of  $G$  constitute a basis of  $H_1(G)$ . Such a basis is called a **fundamental cycle basis** or a **Kirchhoff basis**. When  $G$  is not connected, we can work independently on each connected component of  $G$  since homology is the direct sum of the component homologies. We can even refine this decomposition into 2-connected components (cf. Exercise 4.6).

We will thus assume that  $G$  is connected. When the edges of  $G$  are positively weighted, we can search for a basis that minimizes the sum of the length of its cycles. Such a basis is called a **minimum weight (cycle) basis**. Here, the length of a cycle  $c = \sum_a n_a a$  is  $|c|_w := \sum_a |n_a| w(a)$  where  $w : C \rightarrow \mathbb{Q}_+^*$  is the weight function. Corollary 6.3 below shows that a minimum cycle bases is made of simple cycles. However, as opposed to Proposition 3.3, a minimum weight basis is not always a fundamental cycle basis. The counterexample in Figure 1 is from Hartvigsen and Mardon [HM93]. In

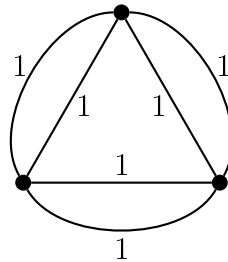


Figure 1: Each spanning tree in this graph is a path of length 2. The corresponding fundamental basis is composed of two cycles of length 2 and two cycles of length 3 leading to a fundamental cycle basis of total weight 10. However a minimum weight basis of total weight 9 is given by the three outer cycles of length 2 and the central triangle.

fact, it seems that little is known concerning the minimum weight bases of the *integer* homology of a graph. Most of the literature on the subject has been concentrated on homology with  $\mathbb{Z}/2\mathbb{Z}$  coefficients. Even in this case, the same counterexample as above shows that a minimum weight basis is not always a fundamental cycle basis. Hartvigsen and Mardon [HM93] characterize the graphs possessing a minimum weight basis that is also a fundamental cycle basis, independently of the weight function.



In general, looking for the minimum weight *fundamental* basis is NP-hard [DPeK82]. However, Horton [Hor87] proved that computing a minimum weight basis with  $\mathbb{Z}/2\mathbb{Z}$  coefficients can be done in polynomial time. We now present his algorithm.

If  $B$  is a family of cycles of  $G$ , we denote by  $\ell(B)$  the list of lengths of the cycle of  $B$  in increasing order. We first observe that

**Lemma 6.1.** *A basis  $B$  of  $H_1(G, \mathbb{Z}/2\mathbb{Z})$  has minimum weight if and only if  $\ell(B)$  is minimal for the lexicographic order. The following algorithm thus returns a minimum weight basis.*

1. Initialize  $B$  to the empty set.
2. Scan the cycles in  $H_1(G, \mathbb{Z}/2\mathbb{Z})$  in increasing order of their length. At each step, add the scanned cycle  $c$  to  $B$  if  $B \cup \{c\}$  is an independent family.
3. return  $B$ .

PROOF. Remark that using the coefficient field  $\mathbb{Z}/2\mathbb{Z}$  provides the homology group  $H_1(G, \mathbb{Z}/2\mathbb{Z})$  with a vector space structure. (It is thus a matroid to which we can apply the classical greedy algorithm.) It is then a simple exercise to show that minimum weight bases are indeed characterized by the minimality of their length list for the lexicographic order.  $\square$

Since  $H_1(G, \mathbb{Z}/2\mathbb{Z})$  contains  $2^{\beta_1(G)}$  cycles, this algorithm is not very efficient. In order to restrict the search, Horton characterizes the cycles that may belong to a minimum weight basis.

**Lemma 6.2.** *Suppose  $b = c + d$  is a cycle of a basis  $B$  of  $H_1(G, \mathbb{Z}/2\mathbb{Z})$ . Then either  $B \setminus \{b\} \cup \{c\}$  or  $B \setminus \{b\} \cup \{d\}$  is a basis.*

PROOF. If  $c$  and  $d$  were both in the linear span of  $B \setminus \{b\}$ , then so would  $b$ .  $\square$

**Corollary 6.3.** *The cycles of a minimum weight basis are simple.*

PROOF. Suppose that  $b$  is a non-simple cycle of a minimum weight basis  $B$ . Then  $b$  can be written as the sum  $b = c + d$  of two edge disjoint cycles. In particular,  $b$  is longer than  $c$  or  $d$ . By the preceding lemma, we can replace  $b$  by  $c$  or  $d$  in  $B$  to get a shorter basis, contradicting the minimality of  $B$ .  $\square$

**Lemma 6.4.** *Let  $b = p \cdot q^{-1}$  be a cycle of a minimum weight basis, where  $p$  and  $q$  are two edge disjoint paths. Then, either  $p$  or  $q$  is a shortest path for  $|\cdot|_w$ .*

PROOF. Let  $r$  be a shortest path from the common initial vertex of  $p$  and  $q$  to their common last vertex. With a little abuse of notation, we can write  $b = p \cdot r^{-1} + r \cdot q^{-1}$ . By Lemma 6.2,  $b$  must be no longer than  $p \cdot r^{-1}$  or  $r \cdot q^{-1}$ , implying that either  $q$  or  $p$  is a shortest path.  $\square$



**Corollary 6.5.** *Let  $v$  be a vertex of a cycle  $b$  of a minimum weight basis. Then  $b = p \cdot a \cdot q^{-1}$ , where  $a$  is an arc and  $p, q$  are two shortest paths with  $v$  as initial vertex.*

PROOF. We write  $b = (a_1, a_2, \dots, a_k)$  with  $v = o(a_1) = o(a_k^{-1})$ . Let  $i$  be the maximal index such that  $(a_1, a_2, \dots, a_i)$  is a (simple) shortest path. Then  $b = (a_1, a_2, \dots, a_i) \cdot a_{i+1} \cdot (a_{i+2}, \dots, a_k)$  and the previous lemma implies that  $(a_{i+2}, \dots, a_k)$  is a (possibly empty) shortest path  $\square$

When there is a unique shortest path between every pair of vertices, this corollary allows us to reduce the scan of step (2) in Lemma 6.1 to  $|V||A_+|$  cycles, one for each (vertex,edge) pair. In general this cannot be assumed<sup>3</sup> and there still might be too many cycles to test. Suppose that for every vertex  $v$  we choose a shortest path tree  $T_v$  with root  $v$ . For every pair  $(v, a) \in V \times A_+$ , let  $c(v, a) = T_v[v, a]$  be the loop with basepoint  $v$  through  $a$  relatively to  $T_v$ .

**Lemma 6.6.** *The scan of step (2) in the greedy algorithm of Lemma 6.1 can be restricted to the loops  $c(v, a)$  for  $(v, a) \in V \times A_+$ .*

PROOF. It is enough to prove (cf. Exercise 6.7 below) that there exists a minimum weight basis composed of cycles of the form  $c(v, a)$  only. Let  $b$  be a cycle of some minimum weight basis. For every vertex  $v$  of  $b$ , Corollary 6.5 gives a decomposition  $b = p \cdot a \cdot q^{-1}$ , where  $p$  and  $q$  are shortest paths with origin  $v$  and  $a$  is an arc. Let  $d_v$  be the number of arcs in  $b$  that are not in  $c(v, a)$ . We define the *default value*  $d(b)$  as the minimum of  $d_v$  taken over all the vertices of  $b$ .

We now consider a minimum weight basis  $B$  minimizing the sum  $\sum_{b \in B} d(b)$  of the default values of its cycles. If this sum is zero, then all the cycles have the form  $c(v, a)$  and we are done. Otherwise, consider a cycle  $b \in B$  that is not equal to any  $c(v, a)$ . Let  $b = p \cdot a \cdot q^{-1}$  be a decomposition for which the minimum  $d(b)$  occurs. Denote by  $x$  and  $y$  the endpoints of  $a$  and by  $v$  the starting vertex of  $p$ , so that  $c(v, a) = T_v[v, x] \cdot a \cdot T_v[y, v]$ . We can write

$$b = p \cdot T_v[x, v] + c(v, a) + T_v[v, y] \cdot q^{-1}$$

Applying Lemma 6.2 twice we see that  $b$  can be replaced in  $B$  by at least one of the three cycles in the above sum, to produce another cycle basis. If  $p \cdot T_v[x, v]$  is shorter than  $b$  or is non-simple, then it cannot replace  $b$  by (weight) minimality of  $B$ . Otherwise, by writing the cycle  $p \cdot T_v[x, v]$  as  $p' \cdot e \cdot T_v[x, v]$  with  $p = p' \cdot e$ , it is easily seen that the default value of this cycle is strictly less than  $d(b)$ . The same is true for  $T_v[v, y] \cdot q^{-1}$ . In any case,  $b$  can be replaced by a cycle whose default value is strictly less than  $d(b)$ . This is in contradiction with our choice of  $B$ .  $\square$

*Exercise 6.7.* Suppose that the cycles of  $B$  all belong to a subset  $C \subset H_1(G)$ . Check that the scan of step (2) in the greedy algorithm of Lemma 6.1 can be restricted to  $C$ .

<sup>3</sup> A probabilistic perturbation [CCE13] technique allows to enforce this assumption, at the price of losing determinism.

**Proposition 6.8.** *A minimum weight basis of  $G$  can be computed in  $O(|V|^2 \log |V| + \beta_1^2(G)|V||A|) = O(|V||A|^3)$  time.*

PROOF. By Lemma 6.6, we restrict the scan step of the greedy algorithm to the cycles  $c(v, a)$  with  $(v, a) \in V \times A_+$ . For each vertex  $v$ , we compute a shortest path tree  $T_v$  in  $O(|V| \log |V| + |A|)$  time using Dijkstra's algorithm. There are  $\beta_1(G)$  cycles of the form  $c(v, a)$ , each of size  $O(|V|)$ . Their computation and storage for all the vertices  $v$  thus requires  $O(|V|(|V| \log |V| + |A| + \beta_1(G)|V|))$  time. They can be sorted according to their length in  $O(\beta_1(G)|V|^2 \log(\beta_1(G)|V|^2))$  time. In order to check if a cycle is independent of the current family of basis elements, we view a cycle as a vector in  $(\mathbb{Z}/2\mathbb{Z})^{A_+}$ . We use Gauss elimination to maintain the current family in row echelon form. This family has at most  $\beta_1(G)$  vectors and testing a new vector against this family by Gauss elimination needs  $O(\beta_1(G)|A|)$  time. The cumulative time for testing independence is thus  $O(\beta_1^2(G)|A||V|)$ . The whole greedy algorithm finally takes

$$O(|V|(|V| \log |V| + |A| + \beta_1(G)|V|) + \beta_1(G)|V| \log(\beta_1(G)|V|) + \beta_1^2(G)|A||V|)$$

time which reduces to  $O(|V|^2 \log |V| + \beta_1^2(G)|V||A|)$  after simplification.  $\square$

Note that the above scan can be further reduced by discarding the loops  $c(v, a)$  that are not simple. We can also decompose a cycle into a combination of a fixed fundamental basis associated to a tree. The decomposition of a cycle is just given by its trace over the chords of that tree. This allows to represent the current family of basis elements by a matrix of size  $\beta_1(G) \times \beta_1(G)$  instead of  $\beta_1(G) \times A_+$ .

The computation of a minimal weight basis is often designated by the acronym MCB (Minimum Cycle Basis problem). Many properties of minimum weight bases and other short cycles are discussed in Gleiss's thesis [Gle01]. This minimal weight basis problem can be recast in the more formal language of matroids, see Golinski and Horton [GH02]. The greedy algorithm as analysed in Proposition 6.8 is not optimal. Further improvements were proposed [KMMP04, KMMP08, MM09]. For integer coefficients the set of  $\mathbb{Z}$ -homology classes do not form a matroid in general. The greedy algorithm cannot be applied anymore. In fact, Kavitha et al. [KLM<sup>+</sup>09] provide an example of a weighted graph whose minimal weight bases for  $\mathbb{Z}/2\mathbb{Z}$  and integer coefficients differ. The status of the computation of a minimal weight  $\mathbb{Z}$ -homology basis is still unknown.

**Open problem:** What is the complexity of the computation of a minimal weight basis for  $\mathbb{Z}$ -homology?

## 7 Coverings, Actions and Voltages

Covering projections are among the most fruitful morphisms when associated to homotopy. They allow to translate topological properties into group properties, leading to surprisingly simple proofs in one of the two fields. Intuitively, a covering of a graph  $G$  is a morphism  $H \rightarrow G$  that is locally an isomorphism. The graphs  $G$  and  $H$  are

respectively called the **base** and the **total space** of the covering. Looking from the base or from the total space provides different ways of describing coverings. This section details those point of views, leading to a classification of coverings. All the material covered here is classical and can be found in textbooks on algebraic topology such as [Mas77]. It was later recast in the realm of graph theory [GT87, BW09].

## 7.1 Coverings

**Definition 7.1.** The **star** of a vertex  $v$  in a graph  $G$  is the set of arcs with origin  $v$ . It is denoted  $\text{Star}(v) = \{a \in A(G) \mid o(a) = v\}$ .

**Definition 7.2.** A **graph covering** is a graph epimorphism  $p : H \rightarrow G$  such that the restriction  $p : \text{Star}(w) \rightarrow \text{Star}(p(w))$  is bijective for all vertex  $w$  of  $H$ . For  $x$  a vertex or arc of the base graph  $G$ , the set  $p^{-1}(x)$  is called the **fiber** above  $x$ .

Figure 2 depicts a graph covering. If  $p : H \rightarrow G$  is a covering and  $\gamma$  is a path in  $G$ ,

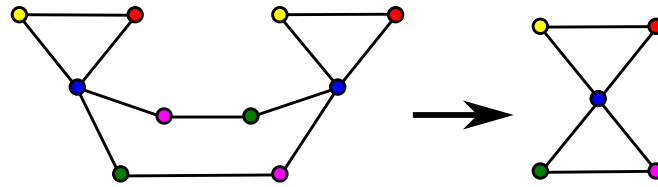


Figure 2: Each vertex of the left graph is sent to the vertex of the same color in the right graph. Arcs are mapped accordingly.

then a path  $\delta$  in  $H$  that projects to  $\gamma$ , i.e., such that  $p(\delta) = \gamma$ , is called a **lift** of  $\gamma$ .

**Lemma 7.3** (Unique lift property). *Let  $w \in V(H)$  with  $p(w) = o(\gamma)$ . There exists a unique lift of  $\gamma$  with origin  $w$ .*

Figure 3 illustrates the property.

**PROOF.** Since  $\text{Star}(w)$  is sent bijectively to  $\text{Star}(o(\gamma))$  by  $p$ , there exists a unique arc in  $\text{Star}(w)$  sent to the first arc of  $\gamma$ . We can continue inductively this way, lifting the arcs of  $\gamma$  one after the other, showing existence and uniqueness of the lift of  $\gamma$  starting from  $w$ .  $\square$

**Lemma 7.4.** *Let  $p : H \rightarrow G$  be a covering. Consider two homotopic paths  $\alpha, \beta$  in  $G$  and two respective lifts  $\tilde{\alpha}$  and  $\tilde{\beta}$  with the same origin. Then  $\tilde{\alpha}$  and  $\tilde{\beta}$  are homotopic in  $H$ .*

**PROOF.** If  $\alpha$  and  $\beta$  are related by one elementary homotopy, then so are  $\tilde{\alpha}$  and  $\tilde{\beta}$  since a spur  $a \cdot a^{-1}$  lifts to a spur. In the general case, the lemma follows by induction on the number of elementary homotopies relating  $\alpha$  to  $\beta$ .  $\square$

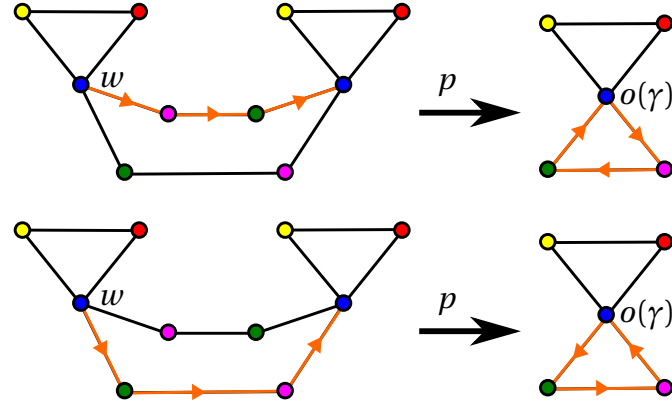


Figure 3: Top, the right closed path with origin  $o(\gamma)$  has a unique lift starting at  $w$ . Bottom, reverting the orientation of the path changes its lift accordingly.

In particular, the final endpoint of the lift of a path  $\alpha$  from a given vertex  $w$  only depends on the homotopy class  $[\alpha]$  of  $\alpha$ . We denote by  $w.[\alpha]$  this final endpoint. We trivially check that for any path  $\beta$  starting at the end of  $\alpha$ :

$$w.[\alpha \cdot \beta] = (w.[\alpha]).[\beta]$$

**Corollary 7.5.** *If  $p : H \rightarrow G$  is a covering, then the induced morphism  $p_* : \pi_1(H, w) \rightarrow \pi_1(G, p(w))$  is one-to-one.*

PROOF. Denote by  $[\alpha]$  the homotopy class of a loop  $\alpha$ . By definition  $p_*[\alpha] = p_*[\beta]$  means  $p(\alpha) \sim p(\beta)$ . By the preceding lemma this implies  $\alpha \sim \beta$ , i.e.,  $[\alpha] = [\beta]$ . In other words  $p_*$  is one-to-one.  $\square$

A direct application of this corollary to the graph coverings of Figure 4 shows that a free group over a countable set of elements embeds as a subgroup of the free group over two elements! Corollary 7.5 tells that the fundamental group of the total space can be seen as a subgroup of the fundamental group of the base. The reciprocal is also true.

**Proposition 7.6.** *Let  $v$  be a vertex of the connected graph  $G$ . For every subgroup  $U < \pi_1(G, v)$ , there exists a connected covering  $p_U : (G_U, w) \rightarrow (G, v)$  with  $p_{U*}\pi_1(G_U, w) = U$ .*

PROOF. Fix a spanning tree  $T$  of  $G$ . We write  $\gamma_a$  for the loop  $T[v, a]$ . Define  $G_U$  by

- $V(G_U) = V(G) \times \{Ug\}_{g \in \pi_1(G, v)}$ ,
- $A(G_U) = A(G) \times \{Ug\}_{g \in \pi_1(G, v)}$ ,
- $o(a, Ug) = (o(a), Ug)$  and  $(a, Ug)^{-1} = (a^{-1}, Ug[\gamma_a])$ ,

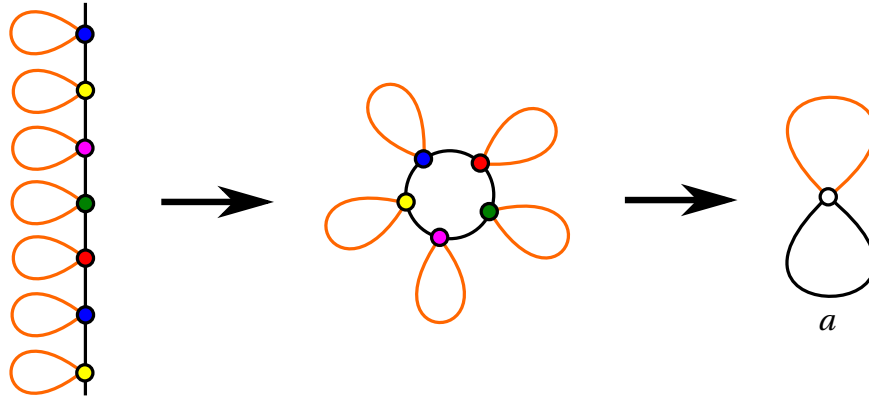


Figure 4: Left, an infinite graph with a countable set of generators. This graph covers the middle graph by mapping vertices and edges according to their colors. The middle graph covers the bouquet  $B_2$  to the right. It follows that the fundamental group of the left graph, a free group over an infinite countable set of elements, embeds into the free group with  $n > 2$  elements which itself embeds into  $F(2)$ .

where  $Ug$  denotes the right coset representative in  $\pi_1(G, v)$  of  $g$  with respect to  $U$ . Schematically, the typical edge of  $G_U$  is

$$(o(a), Ug) \bullet \begin{matrix} \xrightarrow{(a, Ug)} \\ \xleftarrow{(a^{-1}, Ug[\gamma_a])} \end{matrix} \bullet (o(a^{-1}), Ug[\gamma_a])$$

and let  $p_U$  be the projection on the first component. Note that for a vertex  $x$  of  $G$ ,  $\text{Star}(x, Ug) = \text{Star}(x) \times \{Ug\}$ . It follows that  $p_U : \text{Star}(x, Ug) \rightarrow \text{Star}(x)$  is a bijection and that  $p_U$  is indeed a covering.

Let  $\lambda = (a_1, a_2, \dots, a_k)$  be a path from  $v$  to a vertex  $x$  in  $G$ . Setting  $w = (v, U)$ , a simple induction on  $k$  shows that the lift of  $\lambda$  from  $w$  has destination  $w.[\lambda] = (x, U[\gamma_{a_1}] \cdot [\gamma_{a_2}] \cdots [\gamma_{a_k}])$ . In particular, this destination is  $(x, U)$  when  $\lambda$  is contained in  $T$  (see Figure 5, Left) and  $(x, U[\lambda])$  when  $\lambda$  is a loop with homotopy class  $[\lambda] \in \pi_1(G, v)$ . Now, for a vertex  $(x, U[\lambda])$  of  $G_U$ , we have  $w.[\lambda \cdot T[v, x]] = (x, U[\lambda])$  (see Figure 5, Right). It

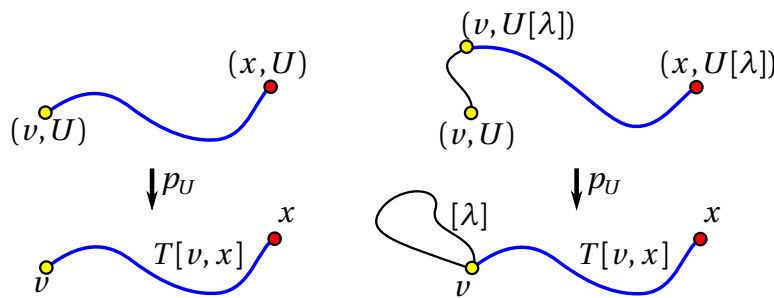


Figure 5: Left and right: two lifts in  $G_U$  of paths in  $G$ .

ensues that  $G_U$  is connected. Finally, a loop  $\lambda$  with basepoint  $v$  satisfies  $[\lambda] \in \text{Im } p_*$  if and only its lift starting from  $w$  is closed, i.e.,  $(v, U[\lambda]) = (v, U)$ . In turns, this means  $[\lambda] \in U$ .  $\square$

*Example 7.7.* If  $G$  is a 2-circuit and  $U = 2\mathbb{Z} < \mathbb{Z} \simeq \pi_1(G, v)$ , we obtain a covering by a 4-circuit as on Figure 6.

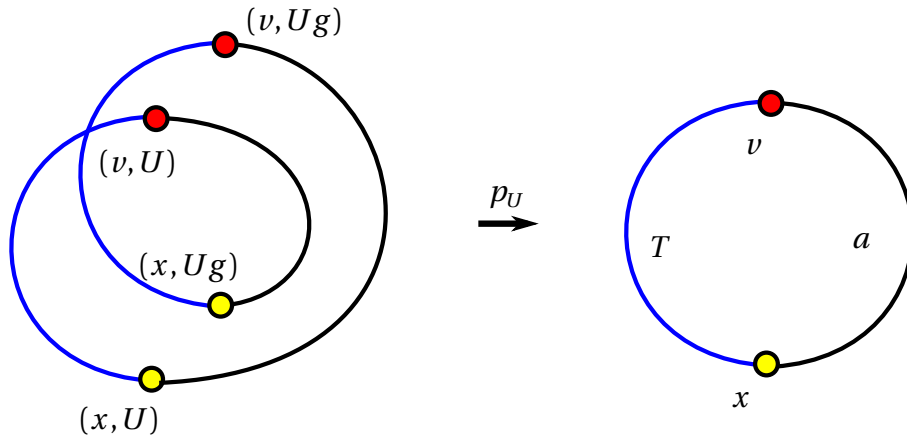


Figure 6: A covering of a 2-circuit. The spanning tree  $T$  is composed of a single edge. The fundamental group  $\pi_1(G, v)$  is generated by  $g = [\gamma_a]$ , so that  $U = \langle g^2 \rangle$ .

As an immediate application, we get

**Theorem 7.8** (Nielsen-Schreier, mid 1920's). *Every subgroup of a free group is free.*

PROOF Let  $F(S)$  be a free group over  $S$ . We realize  $F(S)$  as the fundamental group of the bouquet of circles with edge set  $S$ . By Proposition 7.6, every subgroup of  $F(S)$  is the fundamental group of a graph (covering) which we know to be free<sup>4</sup>.  $\square$

*Exercise 7.9.* Let  $p : H \rightarrow G$  be a graph covering and let  $\alpha$  be a path from a vertex  $v$  of  $H$  to a vertex  $w$  in the same fiber as  $v$ . Show that  $p_*\pi_1(H, w) = [p(\alpha)]^{-1} \cdot p_*\pi_1(H, v) \cdot [p(\alpha)]$ . In particular,  $p_*\pi_1(H, w)$  and  $p_*\pi_1(H, v)$  are conjugate subgroups in  $\pi_1(G, p(v))$ .

### 7.1.1 Covering morphisms

We now consider the set of all the coverings of a given connected graph  $G$ . They can be considered as the objects of a category whose morphisms are defined as follows.

**Definition 7.10.** A **morphism** between coverings  $p : H \rightarrow G$  and  $q : K \rightarrow G$  is a graph morphism  $f : H \rightarrow K$  that sends the fibers to fibers in such a way that the diagram

$$\begin{array}{ccc}
 H & \xrightarrow{f} & K \\
 \searrow p & & \swarrow q \\
 & G & 
 \end{array}$$

is commutative.

Since the restrictions of  $p$  and  $q$  to stars are bijective it must be the case for  $f$ . It follows that  $f$  is a covering. Hence, a covering morphism is a covering of (the total space of) a covering.

<sup>4</sup>Another quick proof uses the fact that a group is free if and only if it acts freely on a tree. But any subgroup obviously acts freely on the same tree, so that it must be free [Ser77].

*Exercise 7.11.* Let  $f$  be a morphism from the covering  $p : H \rightarrow G$  to the covering  $q : K \rightarrow G$ . Consider a vertex  $v$  in  $H$  and a path  $\alpha$  in  $G$  with initial vertex  $p(v)$ . Show the identity

$$f(v).\alpha = f(v.\alpha)$$

*Exercise 7.12.* Show that a covering morphism  $(H, v) \xrightarrow{\quad} (H, v) \xrightarrow{\quad} (G, u)$  with  $H$  connected, must be the identity.

**Lemma 7.13.** *There is a morphism between connected coverings  $p : (H, v) \rightarrow (G, u)$  and  $q : (K, w) \rightarrow (G, u)$  if and only if  $p_*\pi_1(H, v)$  is a subgroup of  $q_*\pi_1(K, w)$  in  $\pi_1(G, u)$ .*

PROOF The condition is clearly necessary. Indeed, if  $f$  is a covering morphism as in the lemma, then by functoriality of  $\pi_1$  it satisfies  $p_* = q_* \circ f_*$ , implying  $p_*\pi_1(H, v) < q_*\pi_1(K, w)$ . It remains to prove that the condition is sufficient. So, we suppose  $p_*\pi_1(H, v) < q_*\pi_1(K, w)$ . We shall construct a covering morphism  $f : H \rightarrow K$ . Let  $x$  be a vertex of  $H$  and let  $\gamma$  be a path from the basepoint  $v$  of  $H$  to  $x$ . We put

$$f(x) := w.[p(\gamma)] = w.p_*[\gamma]$$

If  $a$  is an arc with origin  $x$ , we set  $f(a)$  to the unique edge with origin  $f(x)$  that projects to  $p(a)$  (see Figure 7). We claim that  $f$  is a well-defined map: if  $\lambda$  is another path

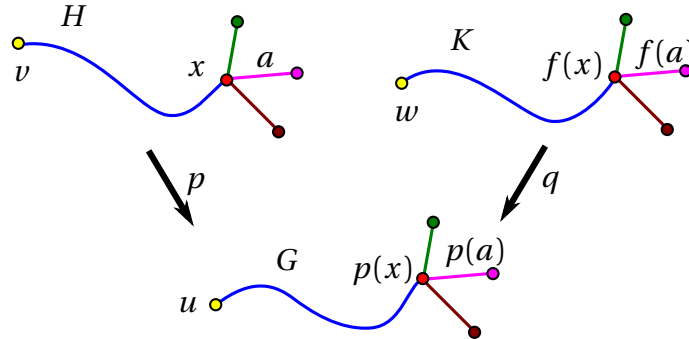


Figure 7: The image of  $x \in V(H)$  is a vertex  $f(x) \in V(K)$  obtained by lifting in  $K$  the projection in  $G$  of a path from  $v$  to  $x$  in  $H$ . Arcs are mapped accordingly.

from  $v$  to  $x$  then  $w.p_*[\lambda] = (w.p_*[\lambda.\gamma^{-1}]).p_*[\gamma]$ . By assumption,  $p_*[\lambda.\gamma^{-1}] \in q_*\pi_1(K, w)$ . This means that the lift of  $p_*[\lambda.\gamma^{-1}]$  from  $w$  is closed, or equivalently:  $w.p_*[\lambda.\gamma^{-1}] = w$ . Whence  $w.p_*[\lambda] = w.p_*[\gamma]$  as claimed. The map  $f$  so defined is clearly a graph morphism: it commutes with the origin and inverse operators. Finally, we have  $q(f(x)) = q(w.p_*[\gamma])$  which is the final endpoint  $p(x)$  of the path  $p(\gamma)$ . Moreover,  $q(f(a)) = p(a)$  by construction, so that  $p = q \circ f$  as required.  $\square$

**Corollary 7.14.** *The connected coverings  $p : H \rightarrow G$  and  $q : K \rightarrow G$  are isomorphic if and only if  $p_*\pi_1(H, v)$  and  $q_*\pi_1(K, w)$  are in the same conjugacy class in  $\pi_1(G, u)$  for  $p(v) = q(w) = u$ .*



PROOF. The condition is necessary by the previous lemma. So, we suppose that  $p_*\pi_1(H, \nu) = g^{-1} \cdot q_*\pi_1(K, w) \cdot g$  for some  $g \in \pi_1(G, u)$ . We easily check that  $g^{-1} \cdot q_*\pi_1(K, w) \cdot g = q_*\pi_1(K, w \cdot g)$  (see Exercise 7.9). It follows that  $p_*\pi_1(H, \nu) = q_*\pi_1(K, w \cdot g)$ , and by two applications of the previous lemma, we get covering morphisms  $(H, \nu) \rightarrow (K, w \cdot g)$  and  $(K, w \cdot g) \rightarrow (H, \nu)$ . By Exercise 7.12 those morphisms are inverse isomorphisms.  $\square$

The corollary reformulates as follows.

**Theorem 7.15.** *The set of isomorphism classes of connected coverings of a connected graph  $G$  corresponds to the set of conjugacy classes of subgroups of the fundamental group of  $G$ . The preorder relation  $K \preceq H$  given by the existence of a covering morphism  $H \rightarrow K$  corresponds to the inclusion  $g^{-1} \cdot \pi_1(H, \nu) \cdot g \subseteq \pi_1(K, w)$  for some  $g \in \pi_1(G, u)$ .*

The trivial group  $\{1\} \subset \pi_1(G, u)$  is obviously the maximal element for this preorder. The corresponding covering is called the **universal cover**. Since its fundamental group is trivial, the universal cover is a tree by Theorem 2.9. Figure 8 shows the universal cover of the Bouquet  $B_2$ .

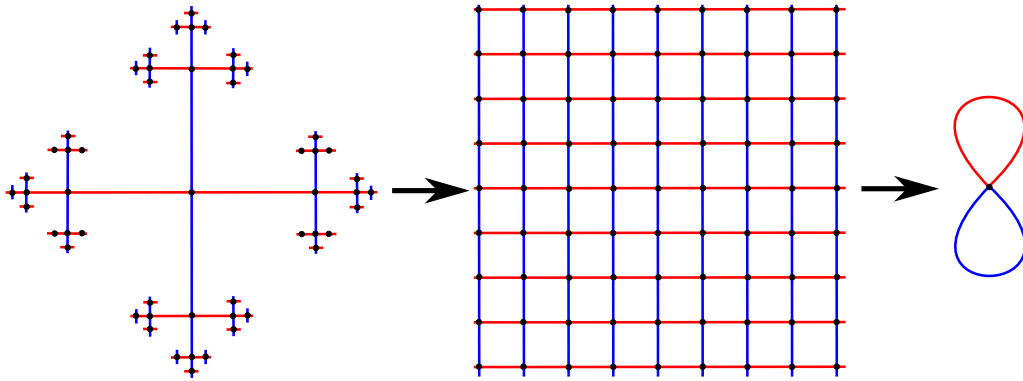


Figure 8: The universal cover of the  $\mathbb{Z}^2$  grid is also the universal cover of  $B_2$ .

## 7.2 Actions and quotients

We denote by  $\text{Aut}(G)$  the group of automorphisms of a graph  $G$ . The **orbit** of a vertex or arc  $x$  of  $G$  by a subgroup  $\Gamma$  of automorphisms is denoted by  $\Gamma \cdot x = \{g(x) \mid g \in \Gamma\}$ .

**Definition 7.16.** The subgroup  $\Gamma < \text{Aut}(G)$  **acts without arc inversion** if for any arc  $a$  of  $G$  and any automorphism  $g$  in  $\Gamma$ , we have  $g(a) \neq a^{-1}$ . In other words  $a^{-1} \notin \Gamma \cdot a$ . If  $\Gamma$  acts without arc inversion, we can define the **quotient graph**  $G/\Gamma$  by

- $V(G/\Gamma) = \{\Gamma \cdot v\}_{v \in V(G)}$ ,
- $A(G/\Gamma) = \{\Gamma \cdot a\}_{a \in A(G)}$ ,
- $o(\Gamma \cdot a) = \Gamma \cdot o(a)$  and  $(\Gamma \cdot a)^{-1} = \Gamma \cdot a^{-1}$

Note that  $\Gamma$  acting without inversions, we have  $(\Gamma \cdot a)^{-1} \neq \Gamma \cdot a$ , i.e., the arc inversion is fixed point free. The **quotient map**  $p_\Gamma : G \rightarrow G/\Gamma$  sending a vertex or arc to its orbit is obviously a graph morphism.

Although the quotient map is onto, it is generally not a covering as illustrated on Figure 9.

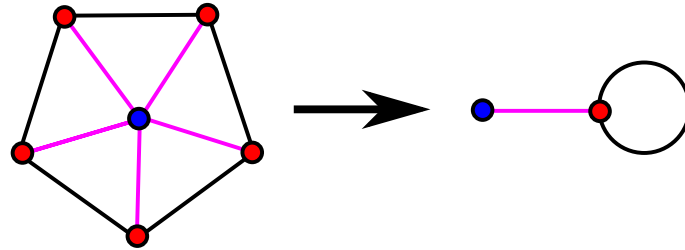


Figure 9: The quotient of the wheel graph with 5 spokes by the subgroup of automorphism generated by the rotation with angle  $2\pi/5$  about the center (blue) vertex. Note that the quotient map is *not* a covering.

**Definition 7.17.** A group of automorphisms  $\Gamma < \text{Aut}(G)$  **acts freely** on  $G$  if it acts without arc inversion and each automorphism in  $\Gamma$  that is not the identity is fixed vertex free (i.e., does not fix any vertex). Intuitively, this means that the corresponding topological (PL) automorphisms (extending the vertex maps to the edges in the obvious way) are fixed point free. Indeed, acting without inversion prevents the automorphisms from fixing the midpoint of edges and being fixed vertex free prevents them from fixing the edge endpoints.

**Proposition 7.18.** *If  $\Gamma$  acts without arc inversion on  $G$ , then  $p_\Gamma : G \rightarrow G/\Gamma$  is a covering if and only if  $\Gamma$  acts freely on  $G$ .*

**PROOF.** Since  $p_\Gamma$  is onto, it is a covering if and only if its restriction to stars is one-to-one. This is equivalent to say that whenever  $a, b$  are two distinct arcs with common origin then  $\Gamma \cdot a \neq \Gamma \cdot b$ . To prove the proposition, we rather show the contrapositive: there exists two distinct arcs  $a, b$  of common origin with the same orbit if and only if there exists a vertex  $v$  fixed by some automorphism  $g \in \Gamma \setminus \{Id\}$ . Indeed, if  $\Gamma \cdot a = \Gamma \cdot b$  then  $a = g(b)$  for some  $g \in \Gamma \setminus \{Id\}$ . This implies  $v = g(v)$  for  $v = o(a)$ . On the other hand, if  $g \neq Id$  fixes a vertex  $v$ , we consider the set of arcs fixed by  $g$ . This set induces a subgraph  $H$  fixed by  $g$ . Since  $g \neq Id$  we have  $H \subsetneq G$  and there must be an arc  $a$  whose origin is in  $H$  but that is not fixed by  $g$ . Then  $a$  and  $b = g(a)$  are two distinct arcs with common origin in the same orbit (see Figure 10).  $\square$

**Lemma 7.19.** *If  $\Gamma$  acts freely on  $G$  then  $(p_\Gamma)_* \pi_1(G, v) \triangleleft \pi_1(G/\Gamma, \Gamma \cdot v)$ .*

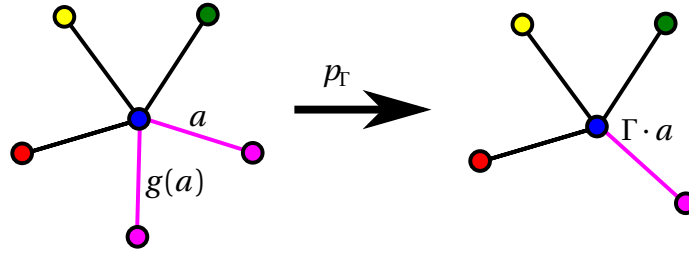


Figure 10: If  $a$  and  $g(a)$  are two distinct arcs with common origin for some  $g \in \Gamma \setminus \{Id\}$  then  $p_\Gamma$  is not a covering.

PROOF. Let  $p_\Gamma(\alpha)$  be a representative of an element in  $(p_\Gamma)_*\pi_1(G, v)$  and let  $\beta$  be a loop with basepoint  $\Gamma \cdot v$  in  $G/\Gamma$ . We just need to show that the conjugate  $\beta \cdot p_\Gamma(\alpha) \cdot \beta^{-1}$  represents a class in  $(p_\Gamma)_*\pi_1(G, v)$ , or equivalently that the lift of  $\beta \cdot p_\Gamma(\alpha) \cdot \beta^{-1}$  starting from  $v$  is a (closed) loop.

Since  $\beta$  is closed in  $G/\Gamma$ , we have  $p_\Gamma(v \cdot \beta) = p_\Gamma(v)$ . This means that  $v \cdot \beta \in \Gamma \cdot v$ , i.e., that there exists  $g \in \Gamma$  with  $g(v) = v \cdot \beta$ . Hence,

$$v \cdot (\beta \cdot p_\Gamma(\alpha) \cdot \beta^{-1}) = g(v) \cdot (p_\Gamma(\alpha) \cdot \beta^{-1}) = (g(v) \cdot p_\Gamma(\alpha)) \cdot \beta^{-1}$$

On the other hand, the lift of  $p_\Gamma(\alpha)$  from  $g(v)$  is  $g(\alpha)$  (see Figure 11) and is thus closed. It follows that  $(g(v) \cdot p_\Gamma(\alpha)) \cdot \beta^{-1} = g(v) \cdot \beta^{-1} = v$ , which was to be proved.  $\square$

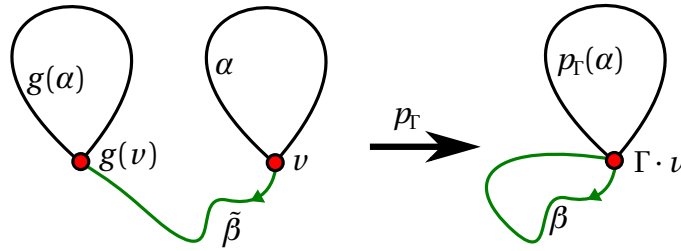


Figure 11: The lift of  $p_\Gamma(\alpha)$  from  $g(v)$  is  $g(\alpha)$ .

**Definition 7.20.** If  $p : H \rightarrow G$  is a graph covering, we denote by  $\text{Aut}(p)$  the group of automorphisms of  $p$ . This is the subgroup of  $\text{Aut}(H)$  composed of the automorphisms

$f$  of  $H$  preserving the fibers of  $p$ , i.e., such that the diagram

$$\begin{array}{ccc} H & \xrightarrow{f} & H \\ & \searrow p & \swarrow p \\ & & G \end{array} \quad \text{com-}$$

mutates. Automorphisms in  $\text{Aut}(p)$  shuffle the vertices in each fiber and are sometimes called **deck transformations** by analogy with the shuffling of a deck of playing cards.

**Lemma 7.21.** If  $H$  is connected,  $\text{Aut}(p)$  acts freely on  $H$ .

PROOF. Let  $f \in \text{Aut}(p)$ . Since  $p(f(a)) = p(a)$  for all arcs  $a$ , we cannot have  $f(a) = a^{-1}$ . For the arc  $p(a)$  would be equal to its inverse  $p(a)^{-1}$ , a contradiction. It follows

that  $\text{Aut}(p)$  acts without arc inversion. On the other hand, suppose that  $f$  fixes some vertex  $v$ . Consider any other vertex  $w$  of  $H$  and a path  $\alpha$  from  $v$  to  $w$ . We compute

$$f(w) = f(v).[p(f(\alpha))] = v.[p(\alpha)] = w$$

So that  $f$  fixes all the vertices. Moreover, the restriction of  $p$  to stars being bijective, the commutation equation  $p(f(a)) = p(a)$  together with  $o(f(a)) = o(a)$  implies  $f(a) = a$ . Consequently,  $f$  must be the identity morphism and the action of  $\text{Aut}(p)$  is fixed vertex free.  $\square$

In conjunction with Proposition 7.18, Lemma 7.21 implies that the quotient projection  $H \rightarrow H/\text{Aut}(p)$  is a covering. It is natural to ask whether this covering is isomorphic to  $p$ . In particular, when  $p$  arises as a quotient, we have

**Lemma 7.22.** *If  $\Gamma < \text{Aut}(G)$  acts freely on a connected graph  $G$ , then  $\text{Aut}(p_\Gamma) = \Gamma$ .*

PROOF. By definition,  $\Gamma \subseteq \text{Aut}(p_\Gamma)$  and  $\Gamma$  acts transitively on the fibers of  $p_\Gamma$ . It is thus enough to prove  $\text{Aut}(p_\Gamma) \subseteq \Gamma$ . Let  $f \in \text{Aut}(p_\Gamma)$ . Fix a vertex  $v$  in  $H$ . Then,  $f(v)$  being in the fiber of  $v$ , the transitive action of  $\Gamma$  implies the existence of  $g \in \Gamma$  with  $g(v) = f(v)$ . Now,  $f \circ g^{-1}$  is an automorphism of  $\text{Aut}(p_\Gamma)$  fixing  $v$ . It must be the identity by the previous lemma, whence  $f = g \in \Gamma$ .  $\square$

**Lemma 7.23.** *Let  $p : (H, v) \rightarrow (G, u)$  be a connected covering and let  $w \in V(H)$  be in the same fiber as  $v$ . Then  $p_*\pi_1(H, w) = p_*\pi_1(H, v)$  if and only if there exists  $f \in \text{Aut}(p)$  such that  $f(v) = w$ .*

PROOF. For the reverse implication, note that  $f_*$  being an isomorphism, we have  $f_*\pi_1(H, v) = \pi_1(H, w)$ . Using that  $p \circ f = p$ , we deduce  $p_*\pi_1(H, v) = p_* \circ f_*\pi_1(H, v) = p_*\pi_1(H, w)$  as desired. For the direct implication, we shall construct  $f$  as in the lemma. For a vertex  $x$  of  $H$  and a path  $\alpha$  from  $v$  to  $x$ , we set

$$f(x) = w.[p(\alpha)]$$

See Figure 12.  $f$  is well-defined. Indeed, if  $\beta$  is another path from  $v$  to  $x$  then  $\beta \cdot \alpha^{-1}$

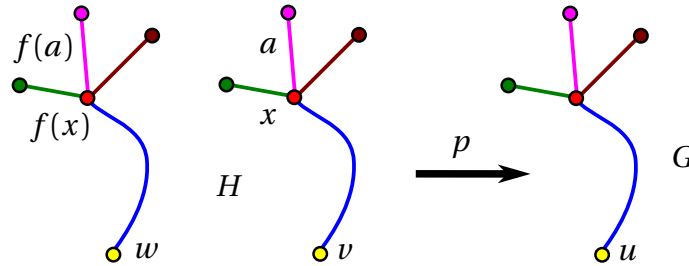


Figure 12: To define  $f(x)$ , we “translate” to  $w$  the origin of a path from  $v$  to  $x$ .

is a loop with basepoint  $v$ . We have  $[p(\beta \cdot \alpha^{-1})] \in p_*\pi_1(H, v) = p_*\pi_1(H, w)$ . It follows that the lift of  $p(\beta \cdot \alpha^{-1})$  from  $w$  is closed. We can thus write

$$w.[p(\beta)] = w.[p(\beta \cdot \alpha^{-1})][p(\alpha)] = w.[p(\alpha)]$$

We can easily extend  $f$  to arcs in order to define a  $p$ -automorphism. The details are left to the reader.  $\square$

**Corollary 7.24.** *If  $p : (H, v) \rightarrow (G, u)$  is a connected covering with  $p_*\pi_1(H, v) \triangleleft \pi_1(G, u)$  then  $\text{Aut}(p)$  acts transitively on the fiber of  $v$ .*

PROOF. Let  $w$  be any vertex in the fiber of  $v$ . We remark that  $p_*\pi_1(H, v)$  being normal in  $\pi_1(G, u)$ , we have  $p_*\pi_1(H, w) = p_*\pi_1(H, v)$  (see Exercise 7.9). The previous lemma allows to conclude.  $\square$

*Exercise 7.25.* With the assumptions of the lemma show that  $\text{Aut}(p)$  acts transitively on any fiber, not just the fiber of  $v$ .

**Proposition 7.26.** *Let  $p : (H, v) \rightarrow (G, p(v))$  be a connected covering and let  $\Gamma < \text{Aut}(H)$  be a subgroup of automorphisms of  $H$  acting without arc inversion. Then,  $p$  and  $p_\Gamma$  are*

*isomorphic, i.e., there is an isomorphism  $H/\Gamma \rightarrow G$  making the diagram*

$$\begin{array}{ccc} & H & \\ p_\Gamma \swarrow & & \searrow p \\ H/\Gamma & \xrightarrow{\simeq} & G \end{array}$$

*commute, if and only if*

1.  $\Gamma = \text{Aut}(p)$ , and
2.  $p_*\pi_1(H, v) \triangleleft \pi_1(G, p(v))$

PROOF. Condition (1) is necessary: if  $p_\Gamma$  is a covering then  $\Gamma$  acts freely on  $H$  by Lemma 7.18. Lemma 7.22 then states that  $\Gamma = \text{Aut}(p_\Gamma)$ . In turn, we have  $\text{Aut}(p_\Gamma) = \text{Aut}(p)$  by the commutativity of the diagram in the lemma. So that  $\Gamma = \text{Aut}(p)$  as claimed. Condition (2) is also necessary: By Lemma 7.19, we have  $p_{\Gamma*}\pi_1(H, v) \triangleleft \pi_1(H/\Gamma, p_\Gamma(v))$  whence  $p_*\pi_1(H, v) \triangleleft \pi_1(G, p(v))$ , again by the commutativity of the diagram in the lemma.

It remains to prove that conditions (1) and (2) are sufficient. By Exercise 7.25, condition (2) implies that  $\text{Aut}(p)$  acts transitively on each fiber of  $p : H \rightarrow G$ . It then follows from condition (1) that  $H/\Gamma = H/\text{Aut}(p) \simeq G$ .  $\square$

**Definition 7.27.** A covering as in the proposition, i.e., such that the fundamental group of the total space is normal in the fundamental group of the base, is called **normal** or **regular** or **Galois**.

In any case,  $p$  being normal or not, we can fully describe  $\text{Aut}(p)$  in terms of  $p_*\pi_1(H, v)$ .

**Proposition 7.28.** *Let  $p : H \rightarrow G$  be a covering with  $H$  connected and let  $v$  be a vertex of  $H$ . Then*

$$\text{Aut}(p) \simeq N(p_*\pi_1(H, v)) / p_*\pi_1(H, v),$$

*where  $N(p_*\pi_1(H, v))$  is the **normalizer** of  $p_*\pi_1(H, v)$ , i.e., the largest subgroup of  $\pi_1(G, p(v))$  containing  $p_*\pi_1(H, v)$  as a normal subgroup. In particular, if  $p$  is a normal covering then  $\text{Aut}(p) \simeq \pi_1(G, p(v)) / p_*\pi_1(H, v)$ .*

PROOF. Let  $\lambda \in N(p_*\pi_1(H, v))$ . We have from Exercise 7.9 that  $p_*\pi_1(H, v.\lambda) = \lambda^{-1} \cdot p_*\pi_1(H, v) \cdot \lambda = p_*\pi_1(H, v)$ . It follows from Lemma 7.23 that there exists an automorphism  $f_\lambda \in \text{Aut}(p)$  such that  $f_\lambda(v) = v.\lambda$ . By Lemma 7.21 this automorphism is unique and we have a well-defined map  $\varphi : N(p_*\pi_1(H, v)) \rightarrow \text{Aut}(p)$ ,  $\lambda \mapsto f_\lambda$ .

We next compute  $f_{\alpha.\beta}(v) = v.(\alpha.\beta) = (v.\alpha).\beta = f_\alpha(v).\beta$ . On the other hand,  $f_\alpha(v).\beta = f_\alpha(v.\beta) = f_\alpha \circ f_\beta(v)$  (see Exercise 7.11). It follows that  $\varphi(\alpha.\beta) = \varphi(\alpha) \circ \varphi(\beta)$  showing that  $\varphi$  is a group morphism.

We next claim that  $\varphi$  is onto. Indeed, let  $f \in \text{Aut}(p)$ , and let  $\gamma$  be a path from  $v$  to  $f(v)$  in  $H$ . From Exercise 7.9, we have  $p_*\pi_1(H, f(v)) = \lambda^{-1} \cdot p_*\pi_1(H, v) \cdot \lambda$ , where  $\lambda = [p(\gamma)]$ . On the other hand  $p_*\pi_1(H, f(v)) = p_*\pi_1(H, v)$  by Lemma 7.23. It follows that  $\lambda \in N(p_*\pi_1(H, v))$ . Moreover,  $f = \varphi(\lambda)$  by the preceding paragraph.

We finally note that

$$\ker \varphi = \{\alpha \in N(p_*\pi_1(H, v)) \mid v.\alpha = v\} = p_*\pi_1(H, v)$$

We conclude as desired that  $\text{Aut}(p)$  is isomorphic to the quotient  $N(p_*\pi_1(H, v)) / p_*\pi_1(H, v)$ .  $\square$

### 7.2.1 The monodromy group

Let  $p : (H, v) \rightarrow (G, u)$  be a covering with  $H$  connected. We denote by  $\mathcal{S}_u$  the symmetric group on the fiber  $\mathcal{F}_u = p^{-1}(u)$ . The map  $w \mapsto w.\alpha$ , with  $w \in \mathcal{F}_u$  and  $\alpha \in \pi_1(G, u)$ , defines a right action of  $\pi_1(G, u)$  on  $\mathcal{F}_u$ , called the **modromy action**. Its image in  $\mathcal{S}_u$  is the **monodromy group** of  $p$ . Since  $H$  is connected, the monodromy action is transitive: For every  $w \in \mathcal{F}_u$  we have  $w = v.p(\gamma)$ , where  $\gamma$  is any path from  $v$  to  $w$ . The **stabilizer** of  $v$  for this action is the subgroup of loops  $\alpha \in \pi_1(G, u)$  such that  $v.\alpha = v$ . This is precisely the characterization of elements in  $p_*\pi_1(H, v)$ .

**Lemma 7.29.** *If  $p : (H, v) \rightarrow (G, u)$  is a covering with  $H$  connected, then the restriction  $f \mapsto f|_{\mathcal{F}_u}$  defines a monomorphism  $\text{Aut}(p) \hookrightarrow \mathcal{S}_u$ .*

PROOF. Since a  $p$ -automorphism permutes the elements of  $\mathcal{F}_u$ , the lemma is a simple consequence of Lemma 7.21.  $\square$

## 7.3 Voltage Graphs

Voltage graphs provide a concise way to encode a graph covering by labelling the arcs of the base graph. They were introduced by Gross and Tucker (see [BW09, Ch. 1] for references).

**Definition 7.30.** A **voltage** on a graph  $G$  with values in a group  $\Gamma$  is a map  $\kappa : A(G) \rightarrow \Gamma$  that commutes with the relevant inverse operations:

$$\forall a \in A(G), \quad \kappa(a^{-1}) = \kappa(a)^{-1}$$

When  $\Gamma$  acts on the right on a set  $F$ , the voltage  $\kappa$  induces a covering  $p_\kappa : G_\kappa \rightarrow G$  where  $G_\kappa$  is the graph defined by

- $V(G_\kappa) = V(G) \times F$ ,
- $A(G_\kappa) = A(G) \times F$ ,
- $o(a, s) := (o(a), s)$  and  $(a, s)^{-1} := (a^{-1}, s.\kappa(a))$  for all  $(a, s) \in A(G) \times F$ ,

and  $p_\kappa$  is the projection on the first component,  $(x, s) \mapsto x$ . Schematically, the typical edge of  $G_\kappa$  is

$$(o(a), s) \bullet \begin{array}{c} \xrightarrow{(a,s)} \\ \xleftarrow{(a^{-1}, s.\kappa(a))} \end{array} \bullet (o(a^{-1}), s.\kappa(a))$$

It is a simple matter of definition to check that  $p_\kappa$  is indeed a covering.

*Exercise 7.31.* Give a necessary and sufficient condition on  $\kappa$  and  $\Gamma$  for  $G_\kappa$  to be connected.

In fact, every covering arises this way.

**Lemma 7.32.** *Every covering  $p : H \rightarrow G$  is isomorphic to a covering induced by some voltage on  $G$ .*

PROOF We set  $\Gamma = \pi_1(G, u)$  for some fixed vertex  $u$  of  $G$  and consider the monodromy action of  $\Gamma$  on the fiber  $F = p^{-1}(u)$ , letting  $w.\lambda$  be the final vertex of the lift of  $\lambda \in \Gamma$  starting from  $w \in F$ . We next define  $\kappa(a)$  as the homotopy class of the loop  $T[u, a]$ , for  $T$  a chosen spanning tree of  $G$ . We thus have an induced covering  $p_\kappa : G_\kappa \rightarrow G$ . We shall prove that there exists an isomorphism  $\varphi : G_\kappa \rightarrow H$  making the following diagram commutative:

$$\begin{array}{ccc} G_\kappa & \xrightarrow{\varphi} & H \\ & \searrow p_\kappa & \swarrow p \\ & G & \end{array}$$

To this end, for any two vertices  $x, y$  of  $G$  we introduce a map  $f_x^y : p^{-1}(x) \rightarrow p^{-1}(y)$  between their fibers:

$$\begin{array}{ccc} p^{-1}(x) & \xrightarrow{f_x^y} & p^{-1}(y) \\ w & \mapsto & w.[T[x, y]] \end{array}$$

Note that  $f_x^y$  and  $f_y^x$  are inverse to each other. We next define  $\varphi : G_\kappa \rightarrow H$  by

$$\begin{cases} \forall (x, w) \in V(G) \times F : \varphi(x, w) = f_u^x(w) \\ \forall (a, w) \in A(G) \times F : \varphi(a, w) \text{ is the unique arc with origin } f_u^{o(e)}(w) \text{ above } a \end{cases}$$

and  $\psi : H \rightarrow G_\kappa$  by

$$\begin{cases} \forall v \in V(H) : \psi(v) = (p(v), f_{p(v)}^u(v)) \\ \forall e \in E(H) : \psi(e) = (p(e), f_{p(o(e))}^u(o(e))) \end{cases}$$

It is an easy exercise to check that  $\varphi$  and  $\psi$  are inverse morphisms making the above diagram commute.  $\square$



Fix a vertex  $u$  in  $G$ . A voltage  $\kappa : A(G) \rightarrow \Gamma$  extends to the loops with basepoint  $u$  by defining

$$\kappa(a_1, a_2, \dots, a_k) = \kappa(a_1)\kappa(a_2)\cdots\kappa(a_k)$$

An elementary homotopy on the loop  $(a_1, a_2, \dots, a_k)$  leaves this value unchanged, so that  $\kappa$  induces a group morphism  $\bar{\kappa} : \pi_1(G, u) \rightarrow \Gamma$ .

**Proposition 7.33.** *A covering  $p : H \rightarrow G$  is normal if and only if it is induced by a voltage  $\kappa$  on  $G$  with values in a group  $\Gamma$  acting on itself by right translations. Here, it is assumed that the induced morphism  $\bar{\kappa} : \pi_1(G, u) \rightarrow \Gamma$  is onto for some fixed vertex  $u$  of  $G$ . Otherwise we can still replace  $\Gamma$  by the range of  $\bar{\kappa}$ .*

Note that requiring  $\Gamma$  to act on itself is equivalent to require that  $\Gamma$  acts freely and transitively.

PROOF We first assume that we are given a voltage as in the proposition. Considering the basepoint  $(u, 1_\Gamma)$  in  $G_\kappa$  we easily check that

$$(u, 1_\Gamma).\lambda = (u, \bar{\kappa}(\lambda)) \quad (1)$$

It follows that  $p_{\kappa_*}\pi_1(G_\kappa, (u, 1_\Gamma)) = \ker \bar{\kappa}$  (the set of homotopy classes with closed lift). It ensues that  $p_{\kappa_*}\pi_1(G_\kappa, (u, 1_\Gamma))$  is normal in  $\pi_1(G, u)$ , i.e., that  $p_\kappa$  is a normal covering. Remark that  $\bar{\kappa}$  being surjective implies with (1) that  $G_\kappa$  is connected.

We now assume given a normal covering  $p : (H, v) \rightarrow (G, u)$ . Let  $T$  be a spanning tree of  $G$ . For every arc  $a$  of  $G$ , lemmas 7.21 and 7.23 imply the existence of a unique automorphism  $f_a \in \text{Aut}(p)$  with  $f_a(v) = v.[T[u, a]]$ . We put  $\Gamma = \text{Aut}(p)$  and  $\kappa(a) = f_a$  and let  $\Gamma$  acts on itself on the right. It remains to check that  $p$  and  $p_\kappa$  are isomorphic coverings. We define  $\varphi : G_\kappa \rightarrow H$  by

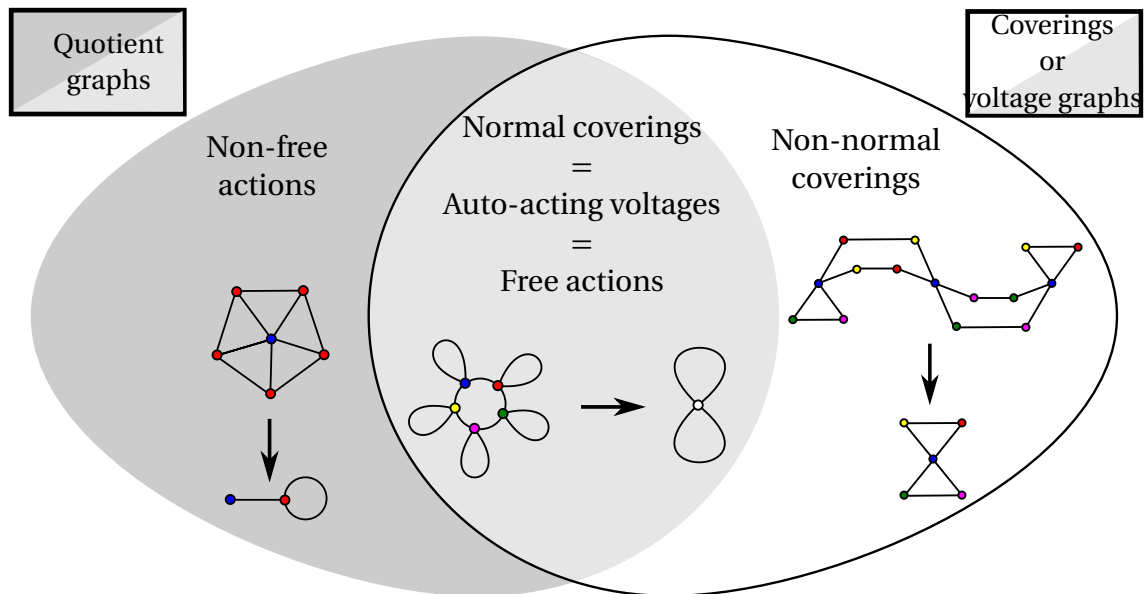
$$\varphi(x, f) = f(v.[T[u, x]])$$

and by extending  $\varphi$  to arcs in the unique way to make it a covering morphism. We also define  $\psi : H \rightarrow G_\kappa$  by

$$\psi(y) = (p(y), f_{v.[T[u, p(y)]]}^y)$$

extending it to arcs. We trivially check that  $\varphi$  and  $\psi$  are inverse morphisms.  $\square$

We end this section on graph coverings with a graphics representing the different types of quotients and coverings.



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