The Homotopy Test

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Contents

A fundamental problem when dealing with curves on surfaces is to decide if a given closed curve can be contracted to a point, or more precisely to a constant curve. This is sometimes referred to as the **contractibility problem**. More generally, we can ask whether two closed curves on a surface are related by a continuous deformation. This question has two variants: we may or may not require the curves to share a given point that remains fixed during the deformation. Note that the problem with fixed point has an obvious reduction to the contractibility problem. Indeed, two curves c, d are homotopic with fixed point if and only if the concatenation $c \cdot d^{-1}$ is contractible. Without the fixed point requirement, that is when the curves are allowed to move *freely* on the surface, the problem is known as the **transformation problem** and can be expressed as a **conjugacy problem**. To see this, choose a point v on a surface S and suppose that c and d are homotopic[1]. We can deform c and d so that each of them passes through p. The resulting curves are still homotopic. In other words, there is a continuous mapping $h : \mathbb{S}^1 \times [0,1] \to S$ such that $h_{|\mathbb{S}^1\times\{0\}} = c$ and $h_{|\mathbb{S}^1\times\{1\}} = d$, and viewing $\mathbb{S}^1 \times [0,1]$ as an annulus, each boundary has a point sent on v by h. We connect these two points by a simple path a in the annulus. The map h sends this path to a closed path α. See Figure 1. Cutting the annulus through a we obtain a disk whose

[1] Homotopy without fixed point is often called *free* homotopy. For concision, we drop the term free. In general, it should be clear from the context whether we use free homotopy or homotopy with fixed point, and we will specify when necessary that the homotopy is with fixed point.

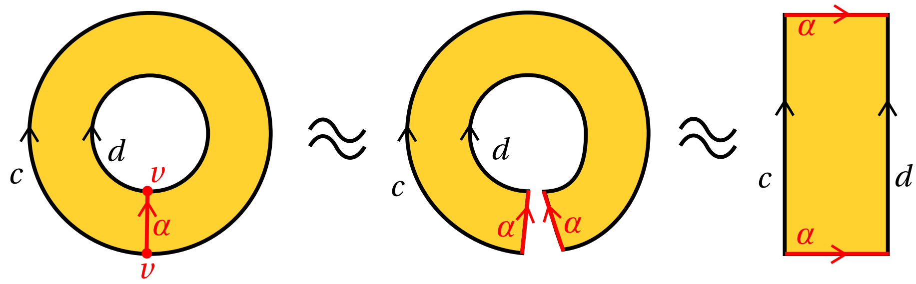

Figure 1: c and d are homotopic if and only if their homotopy classes are conjugate.

boundary is sent to $c \cdot \alpha \cdot d^{-1} \cdot \alpha^{-1}$ which is thus contractible. Hence, c is homotopic to $\alpha \cdot d \cdot \alpha^{-1}$ or, equivalently, the homotopy classes of c and d are conjugate in the fundamental group $\pi_1(S, v)$. For the reverse implication, if c and d have conjugate homotopy classes we can just read Figure 1 from right to left and conclude that c and d are indeed homotopic.

1 Dehn's Algorithm

Suppose that S is a **reduced** combinatorial surface, that is a map with a single vertex and a single face. Its graph G is thus composed of loop edges, each of which corresponds to a generator of the fundamental group of S. We can directly read the homotopy class of a closed path in G: the sequence of arcs of the path translates to the product of the corresponding generators and their inverses. This product is often viewed as a **word** on the generators and their inverses, so that the contractibility problem is the same as the **word problem** where we ask if a product of generators and their inverses is the trivial element in the fundamental group of S.

Max Dehn was among the first to establish and exploit the connection between Topology (the contractibility problem) and Algebra (the word problem). He proposed a solution to the word problem now known as **Dehn's algorithm** [Sti87, paper 5]. Dehn observed that the lift of G in the universal covering space of S induces a tessellation of the plane composed of copies of the unique polygonal face of G in S. This tessellation is actually the **Cayley complex** of $\pi_1(S, v)$ where v is the unique vertex of G. This complex $\tilde{S}$ is relative to the set of generators $\{\beta_i\}_i$ of $\pi_1(S, v)$ – the homotopy classes of the loop edges in G – and to their relation F obtained from the unique facial walk of G in S. The vertex set of $\tilde{S}$ are the elements of $\pi_1(S, v)$ and there is an (oriented) edge labelled β_i between every $\alpha \in \pi_1(S, v)$ and $\alpha \cdot \beta_i$. Finally, disks are glued along each closed path labelled by F in the resulting graph. If a closed path c in G is contractible in S, then any of its lifts is a closed path in $\tilde{S}$. Dehn further claims that
*any closed path in $\tilde{S}$ contains either a **spur**, i.e. an arc followed by its opposite arc, or more than half of F, i.e. a subpath labelled by some word U such that for some other V shorter than U, the concatenation UV is a cyclic permutation of F or its inverse.*
In both cases c is homotopic to a shorter closed path obtained by removing the spur in the former case and by replacing the path labelled by U with the complementary path labelled by V^{-1} in the latter case. This leads to an algorithm where we inductively search for spurs or large pieces of F until we obtain a word that we cannot reduce

anymore. It then follows from Dehn's claim that c is contractible if and only if this word is empty.

In order to prove his claim, Dehn notes that the faces of the complex $\tilde{S}$ are arranged in rings of faces $R_1, R_2, \ldots$, where R_1 is the set of faces incident with a given vertex[2] v_0 of $\tilde{S}$ and R_{i+1} is the set of faces not in R_i sharing a vertex with the external boundary of R_i. Remark that a face of R_{i+1} has at most two vertices in R_i. Hence, if S is an orientable surface of genus $g \geq 2$, each face has $4g$ sides and a face of a ring has at least $4g-2 > 2g$ vertices on its external boundary. Consider now a closed path $\tilde{c}$ without spurs and passing through v_0. Let i be maximal such that $\tilde{c}$ contains a vertex of the external boundary of R_i. Figure 2 illustrates a factious case of a relation of length 6. Since $\tilde{c}$

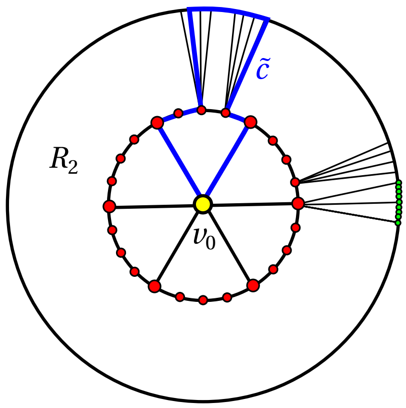

Figure 2: The faces of the complex $\tilde{S}$ are arranged in rings of faces.

has no spurs it is easily seen that it contains the whole intersection of a face with the external boundary of R_i. The previous remark allows to conclude the claim.

Dehn's algorithm has a simple implementation that runs in $O(g|c|)$ time where g is the genus of S. A more careful implementation with $O(g+|c|\log g)$ time complexity was described by Dey and Schipper [DS95]. Finally, optimal $O(g+|c|)$ algorithms were proposed [LR12, EW13]. We shall describe these last approaches to the contractibility and deformation problems, not so far from Dehn's original approach but including more recent techniques borrowed from geometric group theory.

2 van Kampen Diagrams

2.1 Disk Diagrams

A useful tool concerning contractible curves is provided by the so called **van Kampen diagrams**. Such diagrams bear different names in the litterature, among which **disk diagrams** and **Dehn diagrams** are the most common. Intuitively, a disk diagram allows to express the combinatorial counterpart of the following characterization of contractible loops in a topological space X: a loop $\mathbb{S}^1 = \partial\mathbb{D}^2 \to X$ is contractible if and only if it extends to a continuous map $\mathbb{D}^2 \to X$, where $\mathbb{D}^2$ is the unit disk. Given a combinatorial map M with graph G, a **disk diagram over** M is a combinatorial sphere

[2] In his original work, Dehn defines R_1 as a single face.

D with a marked **outer face**, and a labelling of the arcs of D by the arcs of M such that opposite arcs are labelled by opposite arcs and such that every facial walk of D that is not the outer face is labelled by some facial walk of M. In other words, D is a gluing of faces and edges of M that is homeomorphic to the complement of an open disk in a sphere. For instance, this complement could be a tree. In general, it is a tree-like arrangement of topological closed disks connected by trees. The facial walk of the outer face of D is denoted by ∂D. The diagram is **reduced** if any two of its *inner* faces (*i.e.* not the outer one) sharing a vertex v are labelled by facial walks that are not inverse to each other when starting the facial walks at v.

Lemma 2.1 (van Kampen, 1933)**.** *A closed path c in M is contractible if and only if it is the label of the outer facial walk of a reduced disk diagram over M.*

The proof uses the intuitive fact that homotopic closed paths are **combinatorially homotopic**, where a combinatorial homotopy is a sequence of **elementary homotopies** that consist in either inserting or removing a spur, or replacing a subpath of a facial walk by the complementary subpath. See Theorem 4.7 in the previous lecture notes.

PROOF OF LEMMA 2.1. We first prove the existence of a not necessarily reduced disk diagram. Let $c_0 = 1 \to c_1 \to \cdots \to c_k = c$ be a sequence of k elementary homotopies attesting the contractibility of c, where 1 denotes a constant path. By induction on k, we may assume the existence of a disk diagram D such that ∂D is labelled by c_{k-1}. There are three cases to consider.

- If $c_{k-1} \to c_k$ consists in inserting a spur aa^{-1}, then we can form a disk diagram for c_k by attaching a pendant edge labelled with a to the boundary of D.
- If $c_{k-1} \to c_k$ consists in removing a spur, then either this spur corresponds to two consecutive arcs of ∂D with distinct edge support or it corresponds to the two arcs of a single pendant edge. In the former case, we form a disk diagram for c_k by gluing the two arcs along ∂D. In the latter case, we contract the pendant edge.
- Otherwise, $c_{k-1} \to c_k$ consists in the replacement of a subpath p by a subpath q such that pq^{-1} is a facial walk of M. We then perform a subdivision of the outer face of D, inserting a new edge between the extremities of p. The new outer face is chosen among the two new faces as the one not bounded by p. We next subdivide the new edge $k-1$ times, where k is the number of arcs of q. We finally extend the labelling trivially by sending the subdivided edge to the edges of q. This amounts to glue a face with facial walk pq^{-1} along p on D.

If the resulting diagram is not reduced, then there are two facial walks sharing a vertex v and labelled by opposite facial walks of M. We "open" D at v and identify the two facial walks according to the labels of their arcs. This produces a new diagram with two faces less and does not modify the outer face boundary. We repeat the procedure as long as the diagram is not reduced. By induction on the number of faces this procedure must end. Note that the final diagram may have no face, in which case its graph must be a tree corresponding to a closed path that can be reduced to a point by removing spurs only. □

Exercise 2.2. Relates the degree of an inner vertex in a reduced disk diagram over M with the degree of the corresponding vertex in M.

2.2 Annular Diagrams

There is an analogous notion of **annular diagram** defined by a combinatorial sphere with two marked outer faces instead of one.

Lemma 2.3 (Schupp, 1968)**.** *Two closed paths c and d in M are homotopic if and only if there exists a reduced annular diagram over M such that the facial walks of its outer faces (oriented consistently) are labelled with c and d respectively.*

PROOF. By the introductory discussion there exists a path p such that $c \cdot p \cdot d^{-1} \cdot p^{-1}$ is contractible. By Lemma 2.1, there exists a disk diagram over M whose boundary is labelled with $c \cdot p \cdot d^{-1} \cdot p^{-1}$. We may identify the subpaths corresponding to p and p^{-1} respectively and get an annular diagram whose perforated faces are labelled with c and d. If the diagram is not reduced, we proceed as in the proof of Lemma 2.1. □

3 Gauss-Bonnet Formula

Another interesting tool is given by a combinatorial version of the famous Gauss-Bonnet theorem. This theorem relates the curvature of a Riemannian surface S (say a smooth surface embedded into $\mathbb{R}^3$) with its Euler characteristic χ, hence a local geometric quantity with a global topological one. If K is the Gauss curvature of S and k_g is the geodesic curvature along its (smooth) boundary ∂S then:

$$\int_S K \,\mathrm{d}s + \int_{\partial S} k_g \,\mathrm{d}\ell = 2\pi\chi \tag{1}$$

We can obtain a combinatorial version of this formula using some kind of angle structure over a combinatorial surface. Given an orientable combinatorial map $M = (A, \rho, \iota)$, we consider an angular assignment of its corners, that is a real function θ defined over the set of corners. Here, a **corner** is any pair $(a, \rho(a))$, for $a \in A$, of successive arcs around a vertex. We require that the sum of the angular assignments of the corners of any face f satisfies

$$\sum_{c \in f} \theta(c) = d_f/2 - 1, \tag{2}$$

where d_f is the degree of the face, *i.e.* the length of its facial walk. Intuitively, this condition amounts to assume that the faces are Euclidean polygons if we view an angular assignment as a normalized angle, measuring angles in terms of parts of a circle instead of radians. Indeed, the total angle of a Euclidean polygon with d_f sides is $(d_f-2)\pi$, which is $d_f/2-1$ when normalized. We then define the curvature of an interior vertex v as

$$\kappa(v)=1-\sum_{c\in v}\theta(c), \tag{3}$$

where, $c\in v$ indicates that the corner $c=(a,\rho(a))$ is incident to the source vertex v of a. We also define the (geodesic) curvature of a boundary vertex[3] v as

$$\tau(v)=1/2-\sum_{c\in v}\theta(c) \tag{4}$$

Those curvatures thus measure the angle default with respect to the flat situation ($\kappa=1$ and $\tau=1/2$). They can be related to the Gauss curvature of the flat conic surface S_v with one singularity at v obtained by gluing small isocele triangles, one for each corner $c\in v$, with angle $2\pi\theta(c)$ at v. The boundary of S_v is a broken line so that Formula (1) should be corrected with the term $\sum_w(\pi-\alpha_w)$, where w runs over the boundary vertices of S_v and α_w is the interior angle at w. Since the geodesic curvature of a line segment is zero, Formula (1) becomes

$$\int_{S_v} K\,\mathrm{d}s+\sum_w(\pi-\alpha_w)=2\pi\chi=2\pi$$

Noting that with $\sum_w(\pi-\alpha_w)$ is the sum of the angles at the corners of v we obtain $\int_{S_v} K=2\pi\kappa_v$.

Theorem 3.1 (Combinatorial Gauss-Bonnet —)**.** *Let M be a combinatorial map whose boundary is composed of disjoint simple cycles in the graph of M. Denote by χ the Euler characteristic of M and by $V^o\cup V^\partial=V$ its interior and boundary vertex sets. Then, for any angular assignment, we have*

$$\sum_{v\in V^o}\kappa(v)+\sum_{v\in V^\partial}\tau(v)=\chi$$

It is possible to drop the condition on the boundary of M using a slightly different notion of curvature, see Erickson and Whittlesey [EW13]. The above presentation is inspired by Gersten and Short [GS90] and makes the parallel with the differentiable version rather transparent.

PROOF. By definition, we compute

$$\sum_{v\in V^o}\kappa(v)=|V^o|-\sum_{c\in v\in V^0}\theta(c)\quad\text{and}\quad\sum_{v\in V^\partial}\tau(v)=|V^\partial|/2-\sum_{c\in v\in V^\partial}\theta(c)$$

It follows that $\sum_{v\in V^o}\kappa(v)+\sum_{v\in V^\partial}\tau(v)=|V|-|V^\partial|/2-\sum_{c\in v\in V}\theta(c)$. By distributing the corners according to faces rather than vertices and by the angular assignment requirement (2), we see that

$$\sum_{c\in v\in V}\theta(c)=\sum_{c\in f\in F}\theta(c)=\sum_{f\in F}(\frac{d_f}{2}-1)=\frac{1}{2}\sum_{f\in F}d_f-|F|$$

[3]Formally, a combinatorial surface with boundary is defined by marking some faces as perforated, and a boundary vertex is any vertex incident to a perforated face.

where F is the set of faces of M. Since every arc appears in exactly one facial walk, except for those on the boundary of M, we have: $\sum_{f\in F} d_f = 2|E| - |E^\partial|$ where E and E^∂ are the set of edges and boundary edges respectively. Since $|E^\partial| = |V^\partial|$, we conclude that

$$\begin{aligned}\sum_{v\in V^o}\kappa(v)+\sum_{v\in V^\partial}\tau(v) &= |V|-|V^\partial|/2-(|E|-|E^\partial|/2)-|F|) \\ &= |V|-|E|+|F|\end{aligned}$$

□

4 Quad Systems

From an algorithmic point of view it is more convenient to work with combinatorial surfaces all of whose faces are quadrilaterals. We call such a surface a **quadrangulation** or a **quad system**. Given a combinatorial surface without boundary, we easily get a quadrangulation of the same topological surface as follows. We insert a vertex inside each face and connect this vertex to all the corners of the face. Hence, if a facial walk has length k we introduce k new edges in the face. This subdivides each face into triangles. We then delete all the edges of the original graph, thus merging all the triangles by pairs to form quadrilaterals. In practice, we will also require that the vertices have a high degree, say at least 8. For a surface of genus $g \geq 2$ this is easily obtained by first reducing the combinatorial surface to a single vertex and a single face before applying the above quadrangulation process. The resulting quadrangulation has two vertices, $4g$ edges and $2g$ quadrilaterals. Figure 3 shows a reduced surface and its quadrangulation.

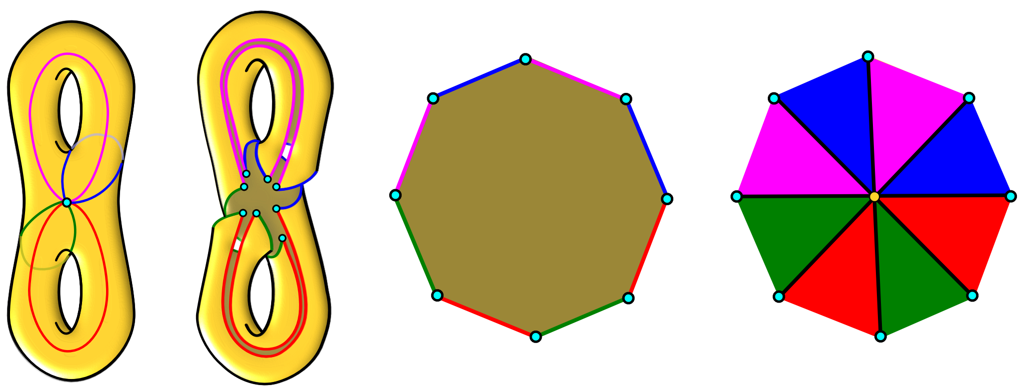

Figure 3: From left to right, a reduced surface is cut-opened and its unique face is triangulated by inserting a vertex in the center. Triangles of the same color are merged by deleting the original loop edges.

Lemma 4.1. *Let Q be a quadrangulation derived by the previous process from a given map M without boundary. We can preprocess M in linear time (proportional to its* *number of arcs) so that any closed walk c can be transformed in $O(|c|)$ time into a homotopic closed walk of size at most $2|c|$ in Q.*

To see this, consider a spanning tree T of the graph G of M. Contracting T gives a surface M' with graph G/T and with a single vertex. Next consider a spanning tree of the dual graph of M' and denote by L the corresponding set of primal edges. Deleting the edges in L leaves a reduced surface M'' and we construct Q by first inserting a new vertex z in the unique face of M'' together with all the edges from z to the corners of the face. We finally remove the remaining edges of G/T to get Q. Note that any edge e of G/T is homotopic to the path of length two in Q connecting z to the two endpoints of e. We can precompute and store these length two paths for each e in total linear time. Now, given any c, we contract all the occurrences of edges of T in c to obtain a homotopic closed walk c' in M'. We further replace every remaining edge by the corresponding length two path to obtain a homotopic closed walk as desired in Q. This transformation takes $O(|c|)$ time.

Exercise 4.2. Propose a construction of quadrangulation starting from a combinatorial surface with nonempty boundary. Can you extend Lemma 4.1 accordingly?

5 Reduction to Canonical form

The last and most important ingredient of the homotopy test is the construction of a canonical representative in each free homotopy class. Given a closed walk in a quadrangulation, the idea is to shorten the walk as much as possible to obtain a combinatorial geodesic. As a homotopy class may contain several geodesics, we further consider the *rightmost* geodesic to define a canonical representative. Once a canonical representative has been computed for two given closed walks we can decide if the walks are homotopic by just checking if their representative are equal up to a circular permutation. The shortening process is based on successive simplifications of spurs and brackets as explained below.

5.1 The Four-Bracket Lemma

Let (a_1, a_2) be a pair of arcs sharing their origin vertex v on a quadrangulation M. Following the terminology of Erickson and Whittlesey [EW13], we define the **turn** of (a_1, a_2) as the number of corners between a_1 and a_2 in counterclockwise order around v. Hence, if v is a vertex of degree d in M, the turn of (a_1, a_2) is an integer modulo d that is zero when $a_1 = a_2$. The **turn sequence** of a subpath $(a_i, a_{i+1}, \dots, a_{i+j-1})$ of a closed walk of length ℓ is the sequence of $j+1$ turns of $(a_{i+k}^{-1}, a_{i+k+1})$ for $-1 \le k < j$, where indices are taken modulo ℓ. The subpath may have length ℓ, thus leading to a sequence of $\ell+1$ turns. Note that the turn of $(a_{i+k}^{-1}, a_{i+k+1})$ is zero precisely when (a_{i+k}, a_{i+k+1}) is a spur. A **bracket** is any subpath whose turn sequence has the form 12^*1 or $\bar{1}\bar{2}^*\bar{1}$ where t^* stands for a possibly empty sequence of turns t and $\bar{x}$ stands for $-x$. Intuitively, if we imagine that every corner of M has a right angle, a bracket corresponds to a straight path ending with right angles. A quadrangulated disk is **non-singular** if its boundary is a simple cycle of its graph.

Lemma 5.1 (Four-bracket —, [GS90, EW13])**.** *Let D be a non-singular quadrangulated disk all of whose interior vertices have degree at least four. Then, the boundary of D contains at least four brackets.*

Figure 4 illustrates the Lemma.

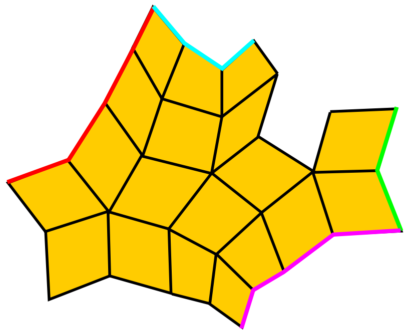

Figure 4: The quadrangulated disk has four highlighted brackets. Can you find them all?

PROOF. Consider the constant angular assignment $1/4$ over D. By the Gauss-Bonnet theorem 3.1, we have $\sum_{v \in int D} \kappa(v) + \sum_{v \in \partial D} \tau(v) = \chi(D) = 1$. By (3), every interior vertex has non-positive curvature. It follows that

$$\sum_{v \in \partial D} \tau(v) \geq 1 \tag{5}$$

Remark that $\tau(v) = (2 - c_v)/4$ where c_v is the number of corners incident to the boundary vertex v. Call v **convex, flat** or **concave** if $c_v = 1$, $c_v = 2$ or $c_v \geq 3$ respectively. In other words v is convex, flat or concave if its curvature is respectively $1/4$, zero or negative. Inequality (5) implies that the boundary of D contains at least four more convex vertices than concave vertices. The lemma easily follows. □

Corollary 5.2. *A nontrivial contractible closed walk in a quadrangulation all of whose interior vertices have degree at least four contains either a spur or a bracket.*

PROOF. Suppose that a nontrivial contractible closed walk c has no spurs. By the van Kampen Lemma 2.1, c is the label of the boundary of a reduced disk diagram D. Let H be the dual graph of D: it has one dual vertex per quadrilateral of D and one dual edge for each pair of quadrilaterals sharing an edge. If H is connected then D is non-singular. Indeed, if the boundary of its outer facial walk ∂D was not a cycle it would contain a degree one vertex, which would contradicts that c has no spurs. We can thus apply the four-brackets lemma 5.1 to conclude that ∂D has at least one bracket. However, the turn t at a vertex of ∂D is the same as the turn of the corresponding vertex in c (up to a multiple of the degree of that vertex in the quadrangulation). It follows that c has also a bracket. If H is not connected, then D consists of a tree-like arrangement of non-singular disks connected by trees through cut vertices. This arrangement has a "degree one" non-singular disk connected to the rest through a single cut vertex. By the previous Lemma 5.1 this disk has four brackets, two of which do not contain the cut vertex. These two brackets thus correspond to brackets in c. □

Exercise 5.3. Show that we can actually claim the existence of a spur or *four* brackets in Corollary 5.2.

5.2 Bracket Flattening

A **bracket flattening** consists in replacing a bracket and the two incident edges with the "straight line" between their endpoints. Some care must be taken when the incident edges of the bracket share their endpoints or when these edges are part of the bracket. Figure 5 depicts the different cases. Corollary 5.2 provides a practical algorithm to test

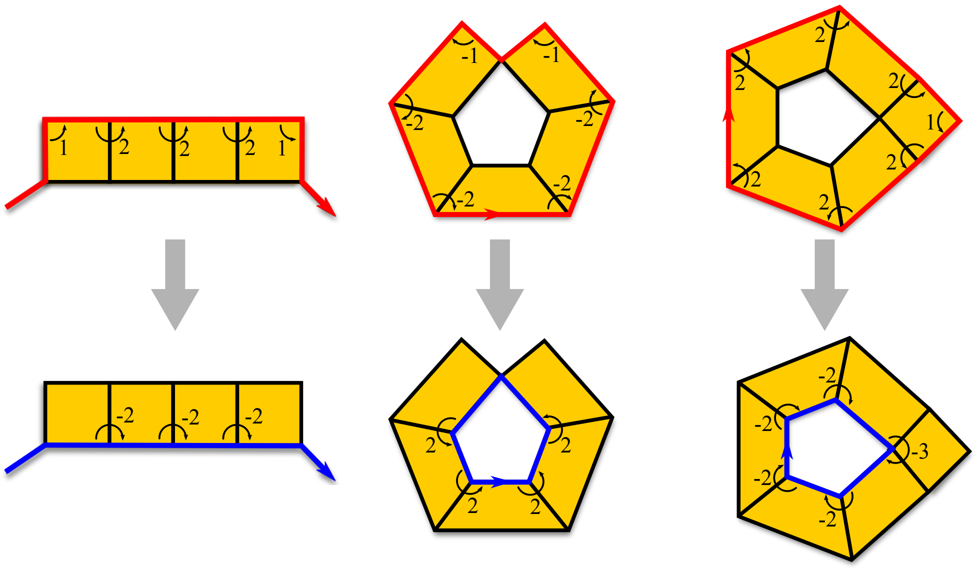

Figure 5: Left, a typical bracket flattening. Middle, the edges incident to the bracket share their endpoints. Right, the bracket covers the whole closed walk.

if a given closed walk c is contractible: remove the spurs and flatten the brackets until there is no more. Then c is contractible if and only if the resulting walk is reduced to a vertex. Note that the non-typical bracket flattening (Figure 5, Right) may only occur when c is non-contractible (why?).

5.3 Canonical representatives

A homotopy class may contain several closed walks without spurs and brackets. In order to get a canonical representative in each homotopy class we further push such reduced walks as much as possible "to their right". Say that a vertex of a walk is **convex** if its turn is 1 in the turn sequence of the walk. If a closed walk c contains a convex vertex v we consider the maximal subpath including v whose turning sequence has the form $x2^*12^*y$, where $x, y \neq 2$. This subpath, say p, bounds an L-shaped sequence of quadrilaterals that lies to its right. Replacing p by the complementary path bounding the sequence of quadrilaterals gives a closed walk homotopic to c with one less convex vertex. Note that this replacement does neither introduce a bracket nor a spur. Some care must again be taken when p covers c. See Figure 6 for all the possible typical and non-typical configurations. A right push reduces the number of convex vertices by

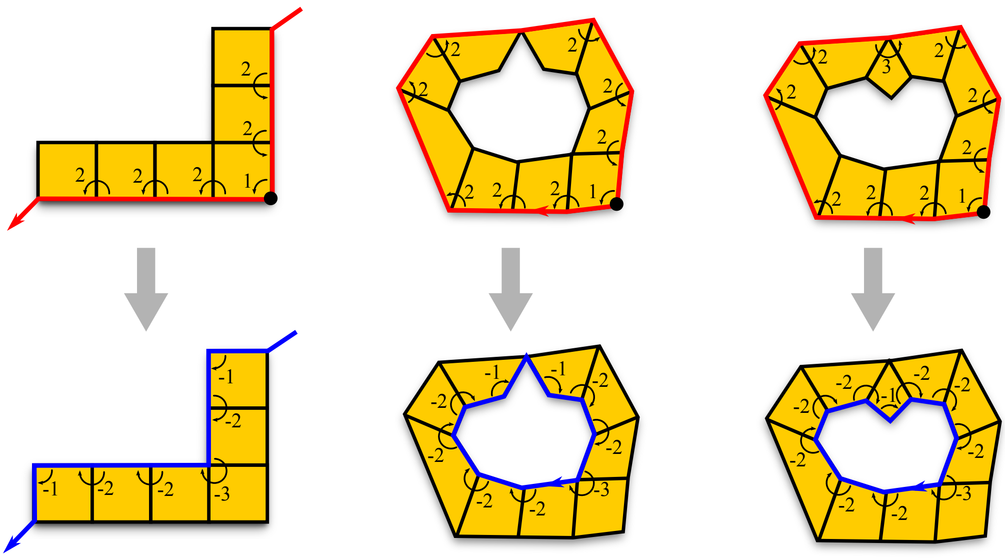

Figure 6: The different configurations for a right push.

one, so that only a linear number of pushes can be applied. A last exceptional case occurs when the turn sequence of c is composed of 2's only. We also apply a right push in this case, which transforms the turn sequence into a sequence of $\bar{2}$ as on Figure 7. When no right pushes apply, the closed walk is said **reduced**, or in **canonical form**.

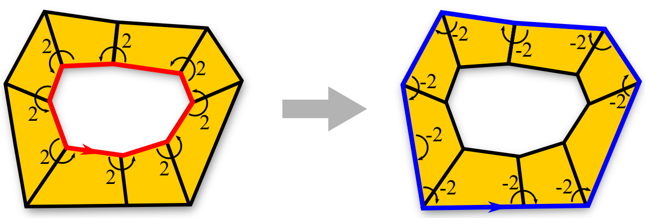

Figure 7: In case all the turns are equal to 2, we push the walk to the right to obtain a sequence $\bar{2}^*$ of turns.

Proposition 5.4. *Let M be a quadrangulation all of whose vertices have degree at least five. Then each homotopy class contains a unique reduced closed walk.*

PROOF. Let c and d be homotopic reduced closed walks. We need to show that $c = d$. Following Lemma 2.3 we consider a reduced annular diagram A for c and d. We first claim that the two boundaries of A are simple. Otherwise, one boundary has a cut vertex that separates A into a smaller annular part A' and a disk part D connected to A' through a single cut vertex. By the four-brackets lemma, the boundary of D has one (in fact at least two) bracket disjoint from this cut vertex. In turn, this bracket would appear in c or d, contradicting the hypothesis that c and d are reduced.

- If the two boundaries of A have a common vertex then cutting through that vertex gives a disk diagram D' bounded by (circular permutations of) c and d. This diagram is a tree-like arrangement of non-singular disks connected by trees through cut vertices. For convenience, we also call cut vertices the two common endpoints of c and d. If a non-singular disk is incident to a single cut vertex, then it is bounded by a subpath of one of c or d. By the four-bracket lemma this subpath would contain a bracket, in contradiction with the reduction hypothesis. It follows that D' is a linear sequence of non-singular disks connected by simple paths (otherwise c or d would have a spur). We claim that none of those non-singular disks can have an interior vertex. Otherwise, considering the constant angular assignment $1/4$ over D', this interior vertex would have negative curvature. An argument similar to the proof of the four-bracket lemma 5.1 shows that the boundary of D' would contain five brackets, one of which not incident to any cut vertex. This would again lead to the contradiction that c or d has a bracket. The dual graph of each non-singular disk is thus a tree. However, no matter the shape of this tree and no matter how its boundary is split one of the resulting boundary paths would contain a bracket or a convex vertex. In both cases this would contradict the fact that c and d are reduced. It follows that D' has no non-singular disk, hence is a simple path, implying that $c = d$.

- Suppose now by way of contradiction that the two boundaries of A are disjoint. The proof of the four-bracket lemma applies to an annulus with a interior vertex of negative curvature to show the existence of a bracket. It follows that as above that A has no interior vertex. Indeed, we can mimic the proof of the four-bracket lemma for an annulus with a vertex of negative curvature to show the existence of a bracket.) The dual graph of A is thus a single cycle with some attached trees. It must actually be a cycle, since otherwise one of the boundaries of A would have a bracket. This cycle has to go straight without bending since otherwise c or d would have a convex vertex or a bracket. (This last case occurs even with a single bend as on Figure 5, Right.) It follows that one of the boundaries of A has 2-turns only as on right Figure 7, contradicting that c and d are reduced. In any case we have reached a contradiction, so that the boundaries of A cannot be disjoint.

□

A reduced closed walk can thus play the role of **canonical representative** for its homotopy class.

6 The Homotopy Test

We now have all the necessary ingredients to perform a linear time homotopy test. Thanks to Lemma 4.1, we can assume given two closed walks in a quadrangulation. We compute the canonical form of each closed walk by first removing spurs and brackets as described in Sections 5.2, then applying right pushes to remove convex vertices as explained in Section 5.3. We claim that this can be done for each closed walk in time proportional to its length.

Theorem 6.1. *The canonical representative of a closed walk c in a quadrangulation, all of whose vertices have degree at least five, can be computed in $O(|c|)$ time.*

PROOF. We first shorten c as mush as possible by removing spurs and flattening brackets incrementally as we traverse c from a chosen basepoint[4]. In order to facilitate the analysis, we consider that each spur amounts to delete two arcs from c and that each bracket flattening amounts to delete its two side arcs and to *move* the edges of the bracket by translating them by one quad. This way, every arc of the final closed walk can be traced back to an arc of c.

We use a stack to store the currently traversed subpath of c and maintain the invariant that this subpath does not contain spurs or brackets. We start with an empty stack corresponding to the path reduced to the basepoint and incrementally push the successive arcs of c. Each time an arc a is pushed on top of the stack we check in constant time[5] if a forms a spur or a bracket with the previous arcs on the stack and update the stack accordingly. We denote by TS the turn sequence of the subpath of c stored in the stack.

- If a forms a spur with the previous arc, *i.e.* if TS has a suffix of length 3 of the form $x0y$, we simply pop a and the previous arc out of the stack. This amounts to replace $x0y$ by the suffix $x+y$ (of length one) in TS.
- If a closes a bracket of length $k+1$, *i.e.* if TS has a suffix of the form $x12^k1y$ or $x\bar{1}\bar{2}^k\bar{1}y$, we pop off the $k+1$ arcs of the (flat part of the) bracket as well as its two incident arcs and push the flat part, translated by one quad, again into the stack. See Figure 5, left. This replaces the suffix of TS by $(x-1)\bar{2}^k(y-1)$ or $(x+1)2^k(y+1)$, respectively.

After this update, the stack clearly contains a path without spurs or brackets. Call a bracket *positive* if its turn sequence has the form 12^*1, and *negative* otherwise.

Claim 1. Suppose that an arc a is moved twice because of two successive bracket flattening, possibly separated by spur removals. Then, these brackets must have the same sign. Moreover, the supports of the brackets, considered as arcs of c, may share at most one arc, this arc being the first arc of the first flattened bracket and the last arc of the second one.

PROOF OF THE CLAIM. Suppose otherwise and assume without loss of generality that the first bracket is positive. For some turn sequence X, TS has the form $Xx12^k1y$

[4] In [EW13], a so-called run-length encoding of the turn sequence of c is used to perform the bracket flattening in linear time. The present algorithm does not use any specific encoding and only uses a stack as a data-structure. This simplification was suggested to us by Saul Schleimer in a Dagstuhl workshop.

[5] E.g., one can maintain a flag to record the fact that turn sequence of the path in the stack ends with 12^* or $\bar{1}\bar{2}^*$.

just before the positive bracket flattening and $X(x-1)\bar{2}^k(y-1)$ just after. When the negative bracket occurs, the turn sequence must have the form $X(x-1)\bar{2}^\ell\bar{1}y'$. Note that $x \neq 0$ since TS does not contain 0 turn after an update. It follows that $x-1=\bar{2}$, so that X must have a suffix of the form $\bar{1}\bar{2}^m$. This would however implies that $Xx12^k1y$ contains a negative bracket (ending with $x=\bar{1}$), a contradiction. See Figure 8, middle.

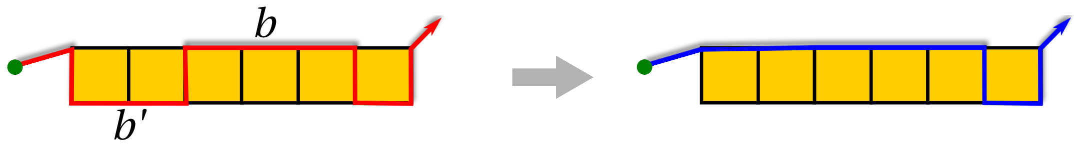

Figure 8: The flattening of the positive bracket b can not be part of a negative bracket, as the smaller bracket b' should first be flattened.

By the preceding paragraph, the successive brackets have the same sign that we assume positive. If the two brackets shared two arcs, then the turn at the vertices in the flat part between these two arcs would be $\bar{2}$ and 2 at the same time. This would imply that they have degree 4, as illustrated in Figure 9, left, in contradiction with the hypohesis of the theorem. Clearly, the shared arc, if any, must occur as the first arc of the first flattened bracket and as the last arc of the second one, or vice-versa. This last case is however impossible as a third bracket would occur in-between the two, and should have been flattened before the second one. See Figure 9, middle. The case

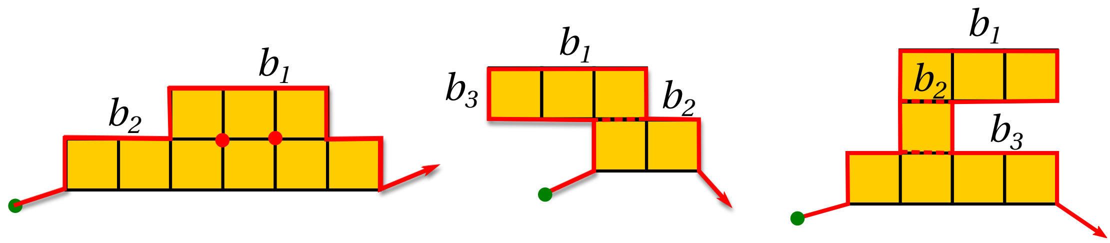

Figure 9: Left, the 3-bracket b_1 share three arcs with the 6-bracket b_2, implying that two (red) vertices have degree four. Middle, if the arc shared by the 3-bracket b_1 and the 2-bracket b_2 is the last arc of b_1, then a bracket b_3 must occur in-between b_1 and b_2. Right, an arc may belong to more than 2 brackets if the intermediate brackets have length 1.

where the brackets are both negative can be dealt with analogously. □

Claim 2. Suppose that an arc a of c is moved thrice because of three successive bracket flattening, possibly separated by spur removals. Then the second bracket is a 1-bracket, *i.e.* has length one.

PROOF OF THE CLAIM. By Claim 1 the three brackets have the same sign that we assume positive. The negative case follows similarly. According to Claim 1, the supports in c of the first and second brackets intersect along a single arc, thus corresponding to a. Moreover a must occur as the last arc of the second bracket. For the same reason, the supports in c of the second and third brackets intersect along the single arc a, occurring as the first arc of the second bracket. It ensues that the second bracket has a single arc. See Figure 9, right. □

In practice, we traverse c twice in order to remove spurs or brackets that would contain the basepoint in their interiors. Some care must be taken to handle the non-typical case as on Figure 5, when a bracket covers the whole of c.

The time spent to traverse c twice is proportional to the number of arcs deletions and moves. According to Claim 2, if an arc is moved several times, then all the brackets that triggered the moves have length one except possibly the first en last ones. Since each 1-bracket corresponds to a single arc of c, it follows that total number of arc moves is bounded by $2|c|$ plus the total number of 1-brackets flattened during the traversal. Now, a bracket flattening decreases the number of edges by two so that their total number is bounded by $|c|/2$. As every arc is morever deleted at most once, we conclude that the number of arc deletions and moves is linear in $|c|$.

Once c has no more spurs or brackets we obtain a geodesic that needs to be pushed to its right in order to remove convex vertices as described in Section 5.3. A right push transforms a subpath of the geodesic into another subpath of the same length without convex vertices. Morover, in the event where a 2-turn appears after a right push, this turn must be surrounded by $\bar{1}$-turns or $\bar{2}$-turns in the resulting turn sequence. See Figure 6, left. It follows that none of the vertices of this subpath will be part of an L-shaped subpath, hence will not be pushed again. We can thus remove all the convex vertices by right pushes using a simple traversal of c without backtracking. Again, some care must be taken to handle the non-typical cases as on Figure 6. The total time needed to obtain a rightmost geodesic is clearly linear. □

Corollary 6.2. *Given two closed walk of length at most ℓ in a combinatorial map of size n we can decide if they are homotopic in $O(n+\ell)$ time.*

PROOF. According to Lemma 4.1, we can reduce the combinatorial map to a quandrangulation in $O(n)$ time and get closed walks homotopic to the given one in $O(\ell)$ time. By Theorem 6.1 we can compute the canonical form of the walks in $O(\ell)$ time. Now these canonical forms, say c and d, are homotopic if and only if one is a circular permutation of the other. This can be tested in linear time by checking whether c is a substring of $d \cdot d$ thanks to the Knuth-Morris-Pratt string searching algorithm [KMP77] [CLRS02, Sec. 32.4]. □

This linear time homotopy test has been implemented and is available as a package of the C++ CGAL library.

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