CR13: Computational Topology
Correction of Exercises #3

Proposition 0.1. *The complete graph K_7 does not embed on the Klein bottle.*

PROOF. Let us denote the vertices of K_7 by $0,1,2,3,4,5,6$. Assume by contradiction that K_7 embeds on the Klein bottle. Then this embedding is a triangulation, by an Euler characteristic argument. Thus 0 is adjacent to six triangles, whose third sides form a 6-cycle. Without loss of generality, this 6-cycle is 123456. Then vertex 1 is surrounded by $602xzy$ (in this order), vertex 2 by $301xwu$, and thus x has to be 4 or 5. Let us first assume that it is 4. Then, looking at the neighborhood of 1 and 6, we see that y has to be 3, and the neighborhoods of the other vertices can be entirely determined in the same manner. On the other hand, if x is 5, we can figure out the rest of the neighborhoods similarly, and we obtain a triangulation that is the same as the previous one (up to relabelling).

This way, we prove that if K_7 triangulates a surface, then this triangulation is fixed (up to relabellings of vertices). In particular, since K_7 triangulates the torus, it can not triangulate the Klein bottle. □